Lepton polarization in neutrino-nucleon interactions

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Why the lepton polarization is interesting?

The atmospheric neutrino data from Kamiokande, IMB, **Super-Kamiokande**, MACRO and SOUDAN 2 strongly support the hypothesis of mixing between **2nd** and **3rd** neutrino generations. It is also supported by the LBL K2K data and does not contradict to the SBL *accelerator* and *reactor* data as well as to the *solar neutrino* experiments.

Experiments Fréjus and NUSEX do not support neutrino oscillations but their statistics is negligible. The result of LSND experiment at Los Alamos is still under discussion.

Short summary

- In the 99.73% C.L., the bounds are: $1.4 \times 10^{-3}\text{eV}^2 < \Delta m^2_{23} < 5.1 \times 10^{-3}\text{eV}^2$ with the best-fit value of $2.6 \times 10^{-3}\text{eV}^2$.
- In the same C.L., the best-fit (effective) mixing is maximal ($\theta_{23} = \pi/4$) with the lower bound given by $\sin^2 2\theta_{23} > 0.86$.
- The fraction $\sin^2\theta_s$ of atmospheric ν_μ's that transform into sterile states ($\nu_\mu \to \cos\theta_s \nu_\tau + \sin\theta_s \nu_s$) is limited by $\sin^2\theta_s < 0.19$ (90% C.L.).

Some examples

- Low and intermediate energy τ leptons generated by neutrinos *inside a detector* ("contained τ events") are *invisible* in water Cherenkov detectors and can be identified only through the secondaries produced in τ decay. The energy and spatial distributions of these secondaries are functionals of the τ polarization.

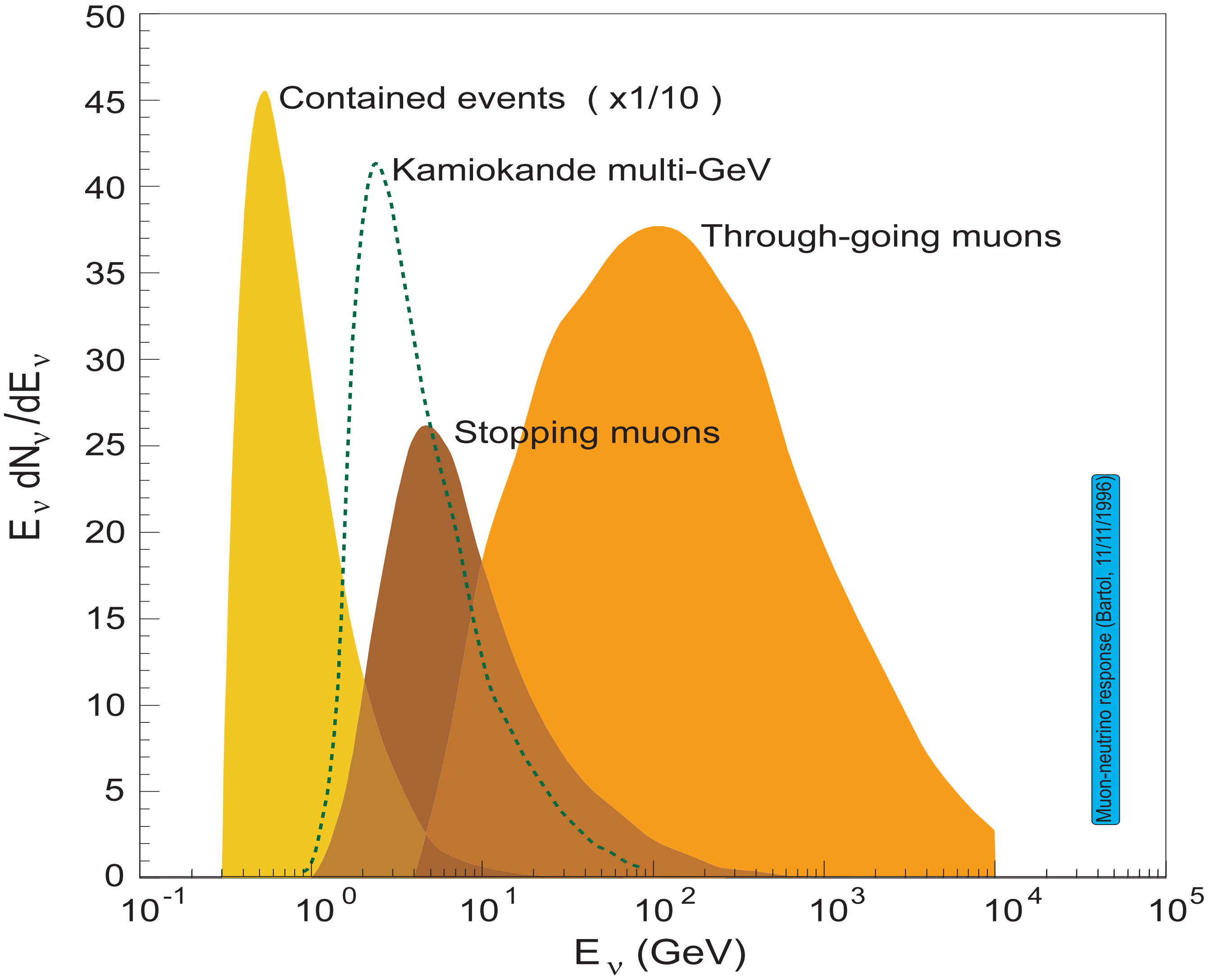
  The contained τ events provide an important signature for $\nu_\mu \to \nu_\tau$ oscillations. Besides they are a source of unavoidable background to the proton decay experiments.

- Consider the chains of reactions *inside the Earth*:

$$\nu_\mu \to \nu_\tau, \quad \nu_\tau + N \to \tau^- + \text{hadrons}, \quad \tau^- \to \nu_\tau + \overline{\nu}_\mu + \mu^-,$$
$$\overline{\nu}_\mu \to \overline{\nu}_\tau, \quad \overline{\nu}_\tau + N \to \tau^+ + \text{hadrons}, \quad \tau^+ \to \overline{\nu}_\tau + \nu_\mu + \mu^+.$$

These chains provide additional contribution into the flux of Throgh-going Upward-going Muons (TUM) and Stopping Muons (SM) which is *absent* in case of $\nu_\mu \to \nu_s$ or $\nu_\mu \to \nu_e$ oscillations. The contribution is small but measurable in future large-scale underground and underwater experiments. The absolute value and energy spectrum of the "$\tau_{\mu 3}$ muons" are dependent of the τ lepton polarization.

$\star$ The relative contribution is energy and zenith-angle dependent.
$\star$ Both energy and angular distributions of the muon charge ratio, μ^+/μ^-, for $\tau_{\mu 3}$ muons are quite differ from ones for "direct" TUM and SM since the longitudinal polarizations of τ^+ and τ^- have opposite signs.



50
45
40
35
30
25
20
15
10
5
0
$E_\nu\, dN_\nu/dE_\nu$
Contained events (x1/10)
Kamiokande multi-GeV
Through-going muons
Stopping muons
Muon-neutrino response (Bartol, 11/11/1996)
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
E_ν (GeV)

- An atmospheric ν_μ or $\overline{\nu}_\mu$ which interact *inside* a detector can produce a muon with energy *below the detection threshold* (Cherenkov threshold for a water detector like Super-Kamiokande). This muon will decay producing an electron. Such an event being classified as "e-like" (for a water detector) or "showering" (for an iron detector) mimics the ν_e ($\overline{\nu}_e$) induced events. Clearly the energy distribution of such events is affected by the muon polarization.

Some relevant projects

- 0.2 kton iron/emulsion detector OPERA
  (Oscillation Project with Emulsion–tRacking Apparatus,
- ICANOE = 5 kton solid target ICARUS (Imaging Cosmic And Rare Underground Signals) + 8 kton solid target *N*^{O}*E* (Neutrino Oscillation Experiment),
- 34 kton magnetized iron detector MONOLITH
  (Massive Observatory for Neutrino Oscillations or LImits on THeir existance),
- 650 kton underground water Cherenkov detector UNO
  (Ultra underground Nucleon detector and neutrino Observatory),
- Multi-megaton water Cherenkov detector TITAND[a]
  (Totally Immersible Tank Assaying Nucleon Decay),

and many others (Hyper-Kamiokande, Aqua-RICH, Super-ICARUS, LANNDD, SCIPIO, ...).

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[a]Former name TITANIC.

The major contribution to the (partially) contained events with a lepton $\ell = e, \mu, \tau$ in final state and to the TUM and SM events comes from the following reactions:

$$\text{QE}: \quad \nu_\ell + n \to \ell^- + p\,, \quad \overline{\nu}_\ell + p \to \ell^+ + n\,,$$

$$\text{RES}: \quad \begin{cases} \nu_\ell + n \to \ell^- + n + \pi^+\,, & \overline{\nu}_\ell + n \to \ell^+ + n + \pi^-\,, \\ \nu_\ell + n \to \ell^- + p + \pi^0\,, & \overline{\nu}_\ell + p \to \ell^+ + p + \pi^-\,, \\ \nu_\ell + p \to \ell^- + p + \pi^+\,, & \overline{\nu}_\ell + p \to \ell^+ + n + \pi^0\,; \end{cases}$$

$$\text{DIS}: \quad \nu_\ell + n \to \ell^- + X\,, \quad \overline{\nu}_\ell + p \to \ell^+ + X\,.$$

Other small but not negligible contributions are from

- coherent CC neutrino scattering [mainly, coherent single pion production, $\nu_\ell(\overline{\nu}_\ell) + A \to \ell^\mp + A + \pi^\pm$, where A is a target nucleus (^{16}O, $^{56}Fe, \ldots$)] and
- strange particle production.

Here we do not consider these processes as well as the nuclear effects. All these are subject of future work.

General formulas

The lepton polarization vector $\mathbf{P} = (P_P, P_T, P_L)$ is defined through the polarization density matrix

$$\left\| \frac{d^2\sigma^{\text{(QE, RES, DIS)}}_{\lambda\lambda'}}{dE_\ell d\cos\theta} \right\| = \frac{1}{2}\left(1 + \boldsymbol{\sigma}\mathbf{P}\right) \frac{d^2\sigma^{\text{(QE, RES, DIS)}}}{dE_\ell d\cos\theta}$$

(the indices $\lambda, \lambda' = \pm 1$ denote the final lepton helicity) whose matrix elements[a] are given by the convolution of leptonic and hadronic tensors:

$$\frac{d^2\sigma_{\lambda\lambda'}}{dE_\ell d\cos\theta} = \frac{G_F^2}{4\pi}\frac{P_\ell}{ME_\nu}\left(1 - \frac{q^2}{M_W^2}\right)^{-2} L^{\alpha\beta}_{\lambda\lambda'} W_{\alpha\beta}\,,$$

$$L^{\alpha\beta}_{\lambda\lambda'}\left(k, k'\right) = j^{\alpha}_{\lambda}\left(k, k'\right) j^{*\beta}_{\lambda'}\left(k, k'\right),$$

$$W_{\alpha\beta} = \frac{1}{4}\int J_\alpha\left(p, p'\right) J^*_\beta\left(p, p'\right)\delta\left(k + k' - p - p'\right)\frac{d\mathbf{p}'}{2p'_0}.$$

[a] Let us omit the marks “QE”, “RES” and “DIS” for simplification.

After standard calculations, we find

$$\frac{d^2\sigma_{++}}{dE_\ell d\cos\theta} = K\left(\frac{E_\ell \mp P_\ell}{M}\right)\left\{(1\pm\cos\theta)\left(W_1 \pm \frac{E_\nu \mp P_\ell}{2M}W_3\right) + \frac{1\mp\cos\theta}{2}\left[W_2 + \frac{E_\ell \pm P_\ell}{M}\left(\frac{E_\ell \pm P_\ell}{M}W_4 - W_5\right)\right]\right\},$$

$$\frac{d^2\sigma_{--}}{dE_\ell d\cos\theta} = K\left(\frac{E_\ell \pm P_\ell}{M}\right)\left\{(1\mp\cos\theta)\left(W_1 \pm \frac{E_\nu \pm P_\ell}{2M}W_3\right) + \frac{1\mp\cos\theta}{2}\left[W_2 + \frac{E_\ell \mp P_\ell}{M}\left(\frac{E_\ell \mp P_\ell}{M}W_4 - W_5\right)\right]\right\},$$

$$\frac{d^2\sigma_{+-}}{dE_\ell d\cos\theta} = K\frac{\sin\theta}{2}\frac{m}{M}\left[\mp\left(2W_1 - W_2 - \frac{m^2}{M^2}W_4 + \frac{E_\ell}{M}W_5\right) - \frac{E_\nu}{M}W_3 - i\frac{P_\ell}{M}W_6\right],$$

$$\frac{d^2\sigma_{-+}}{dE_\ell d\cos\theta} = K\frac{\sin\theta}{2}\frac{m}{M}\left[\mp\left(2W_1 - W_2 - \frac{m^2}{M^2}W_4 + \frac{E_\ell}{M}W_5\right) - \frac{E_\nu}{M}W_3 + i\frac{P_\ell}{M}W_6\right],$$

where the upper (lower) signs are for ν_ℓ ($\overline{\nu}_\ell$) initiated subprocesses and

$$K = \frac{G_F^2 P_\ell}{2\pi}\left(1 - \frac{q^2}{M_W^2}\right)^{-2}.$$

The nucleon structure functions, W_i are defined through the generally accepted representation of the hadronic tensor,

$$\begin{aligned} W_{\alpha\beta} = & -g_{\alpha\beta}W_1 + \frac{p_\alpha\, p_\beta}{M^2}W_2 - \frac{i\,\epsilon_{\alpha\beta\rho\sigma}\, p^\rho\, q^\sigma}{2M^2}W_3 \\ & + \frac{q_\alpha\, q_\beta}{M^2}W_4 + \frac{p_\alpha\, q_\beta + q_\alpha\, p_\beta}{2M^2}W_5 - i\,\frac{p_\alpha\, q_\beta - q_\alpha\, p_\beta}{2M^2}W_6 \end{aligned}$$

and their explicit form is defined by the particular subprocess (QE,RES or DIS).

Therefore the cross section for unpolarized lepton production is

$$\frac{d^2\sigma}{dE_\ell d\cos\theta} = \frac{d^2\sigma_{++}}{dE_\ell d\cos\theta} + \frac{d^2\sigma_{--}}{dE_\ell d\cos\theta} = KR\,,$$

where

$$\begin{aligned} R = & \left(\frac{E_\ell - P_\ell\cos\theta}{M}\right)\left(2W_1 + \frac{m^2}{M^2}W_4\right) + \left(\frac{E_\ell - P_\ell\cos\theta}{m}\right)W_2 \\ & \pm\left[\frac{(E_\nu + E_\ell)\,(E_\ell + P_\ell\cos\theta) - m^2}{M^2}\right]W_3 - \frac{m^2}{M^2}W_5\,. \end{aligned}$$

Finally, the components of the polarization vector, $\mathbf{P}$, are given by

$$P_P = \mp\frac{m\sin\theta}{2MR}\left(2W_1 - W_2 \pm \frac{E_\nu}{M}W_3 - \frac{m^2}{M^2}W_4 + \frac{E_\ell}{M}W_5\right),$$

$$P_T = -\frac{mP_\ell\sin\theta}{2M^2R}W_6,$$

$$P_L = \mp1\pm\frac{2m^2}{M\left(E_\ell+P_\ell\right)R}\left\{\left(2W_1 \pm \frac{E_\nu - P_\ell}{M}W_3\right)\cos^2\frac{\theta}{2}\right.$$
$$\left. + \left[W_2 + \frac{E_\ell+P_\ell}{M}\left(\frac{E_\ell+P_\ell}{M}W_4 - W_5\right)\right]\sin^2\frac{\theta}{2}\right\}.$$

- If $W_6 = 0$ (which is probably true), the vector $\mathbf{P}$ lies in the production plane.
- If $m = 0$ (or $m/E_\ell \ll 1$ and thus $\langle\sin\theta\rangle \ll 1$) then $\ell^\pm$ is fully polarized, $\mathbf{P} = (0, 0, \pm1)$. In particular, for energies of our interest, $e^\pm$ is always fully polarized but, in general, this is not the case for $\mu^\pm$ and $\tau^\pm$.

Quasielastic scattering (QE)

The weak transition currents for ν_ℓ and $\overline{\nu}_\ell$ are defined by

$$\langle p;p'|\widehat{J}^+_\alpha|n;p\rangle = \overline{u}_p\,(p')\,\Gamma_\alpha\,u_n(p) \quad \text{and} \quad \langle p;p'|\widehat{J}^-_\alpha|n;p\rangle = \overline{u}_n\,(p')\,\overline{\Gamma}_\alpha\,u_p(p)$$

with Γ_α written in terms of 6 weak form factors

$$\Gamma_\alpha = \gamma_\alpha F_V + i\sigma_{\alpha\beta}\frac{q_\beta}{2M}F_M + \frac{q_\alpha}{M}F_S + \left(\gamma_\alpha F_A + \frac{p_\alpha + p'_\alpha}{M}F_T + \frac{q_\alpha}{M}F_P\right)\gamma_5.$$

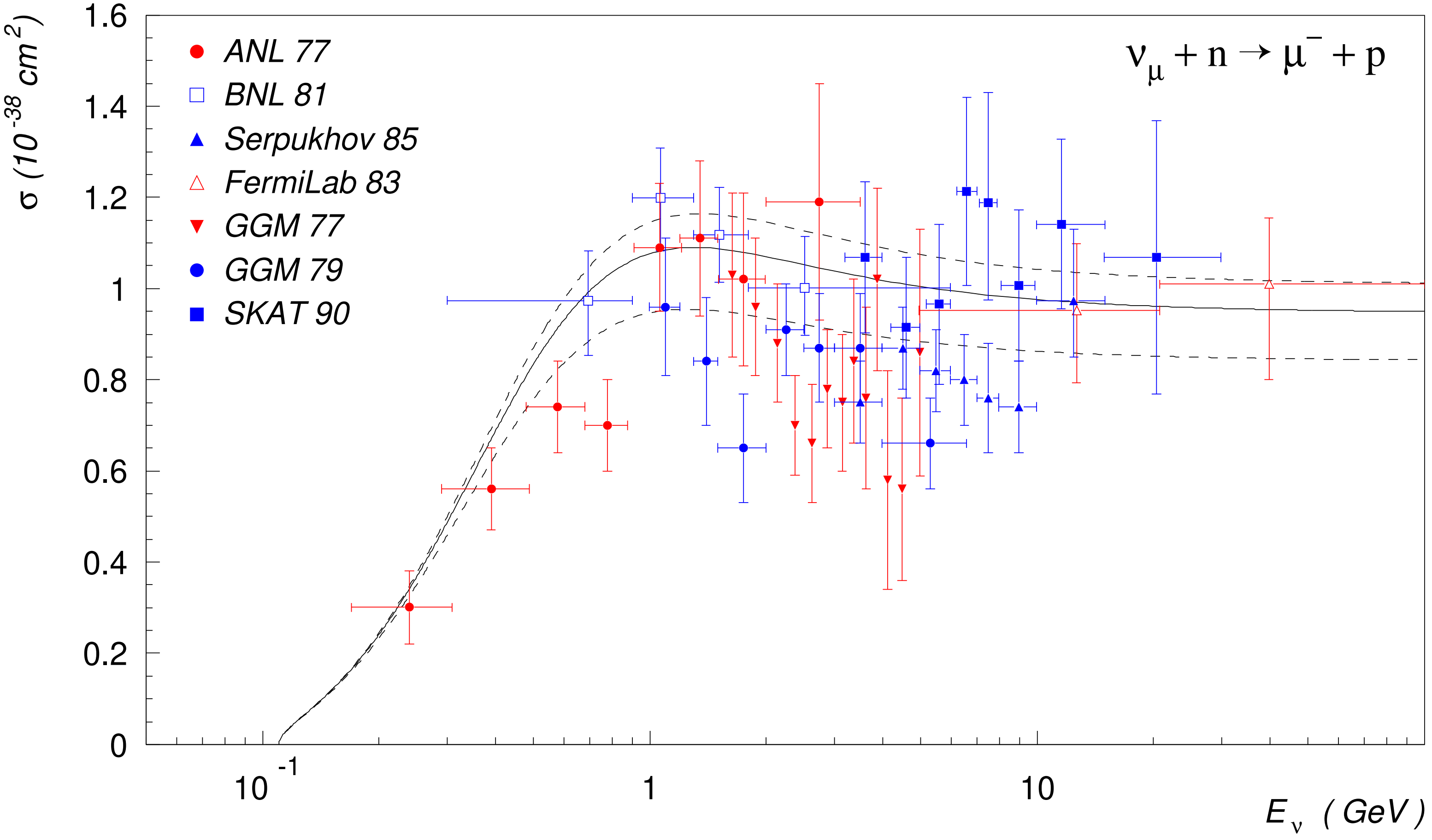
Standard calculations yield

$$W_i^{(\mathsf{QE})} = \cos^2\theta_C\,\frac{M^2}{(pq)}\widetilde{W}_i\,\delta(1-x)\,,$$

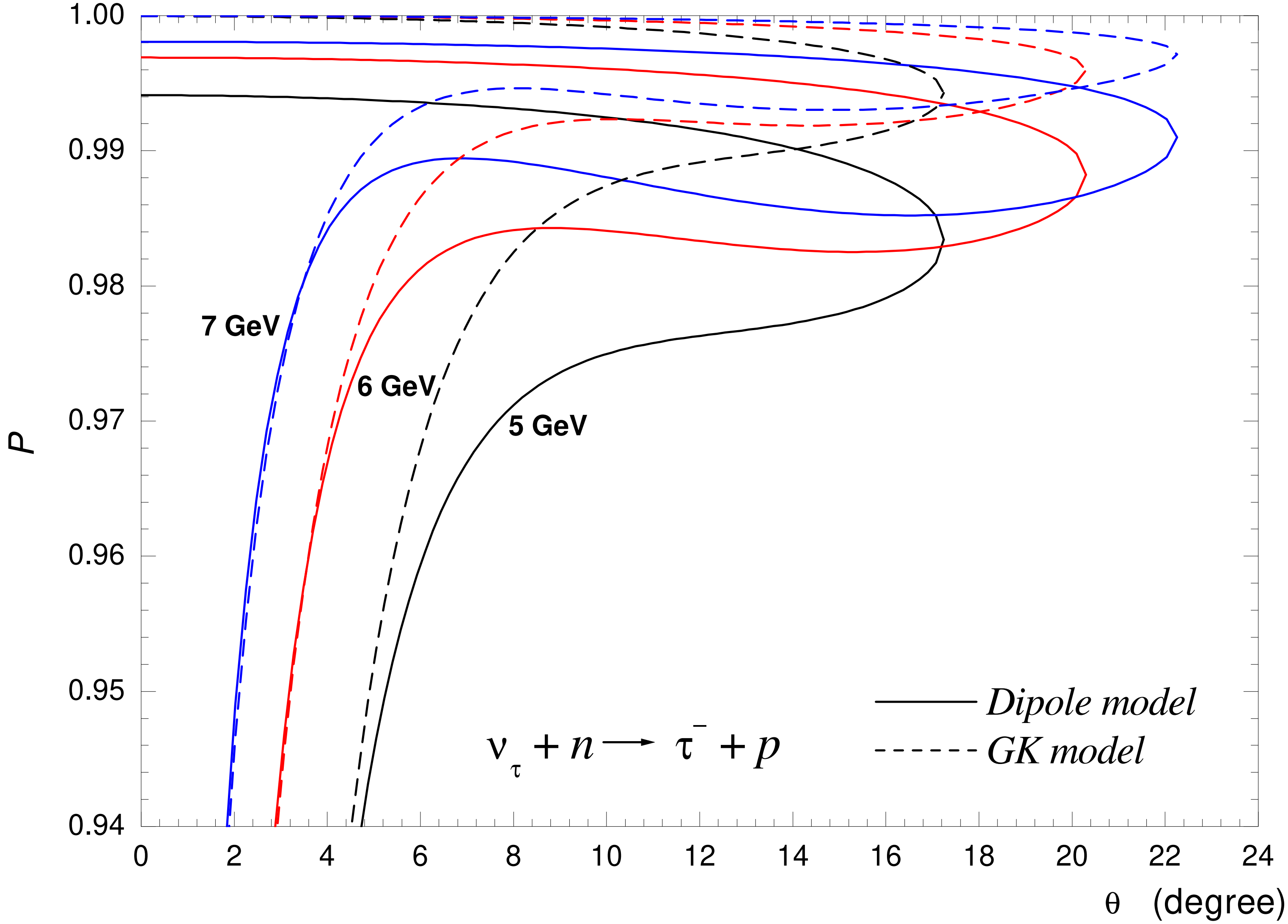
where θ_C is the Cabibbo mixing angle, $x = -q^2/2(pq)$ is the Bjorken scaling variable, and the function $\widetilde{W}_i$ ($i = 1, \ldots, 6$) are the bilinear combinations of the weak form factors.

$$
\begin{aligned}
\widetilde{W}_1 &= F_A^2 - \frac{q^2}{4M^2}\left[F_A^2 + (F_V + F_M)^2\right], \\
\widetilde{W}_2 &= F_A^2 + F_V^2 - \frac{q^2}{4M^2}\left(F_M^2 + 4F_T^2\right), \\
\widetilde{W}_3 &= \, - 2F_A\left(F_V + F_M\right), \\
\widetilde{W}_4 &= F_S\left(F_S + F_V\right) - F_A\left(F_T + F_P\right) - \frac{1}{4}F_M\left(F_M + 2F_V\right) \\
&\qquad - \frac{q^2}{4M^2}\left[\frac{1}{4}F_M^2 + (F_T + F_P)^2 + F_S\left(F_S - F_M\right)\right], \\
\widetilde{W}_5 &= \widetilde{W}_2 + F_S\left(2F_V + \frac{q^2}{2M^2}F_M\right) - F_T\left(2F_A + \frac{q^2}{M^2}F_P\right), \\
\widetilde{W}_6 &= 0.
\end{aligned}
$$

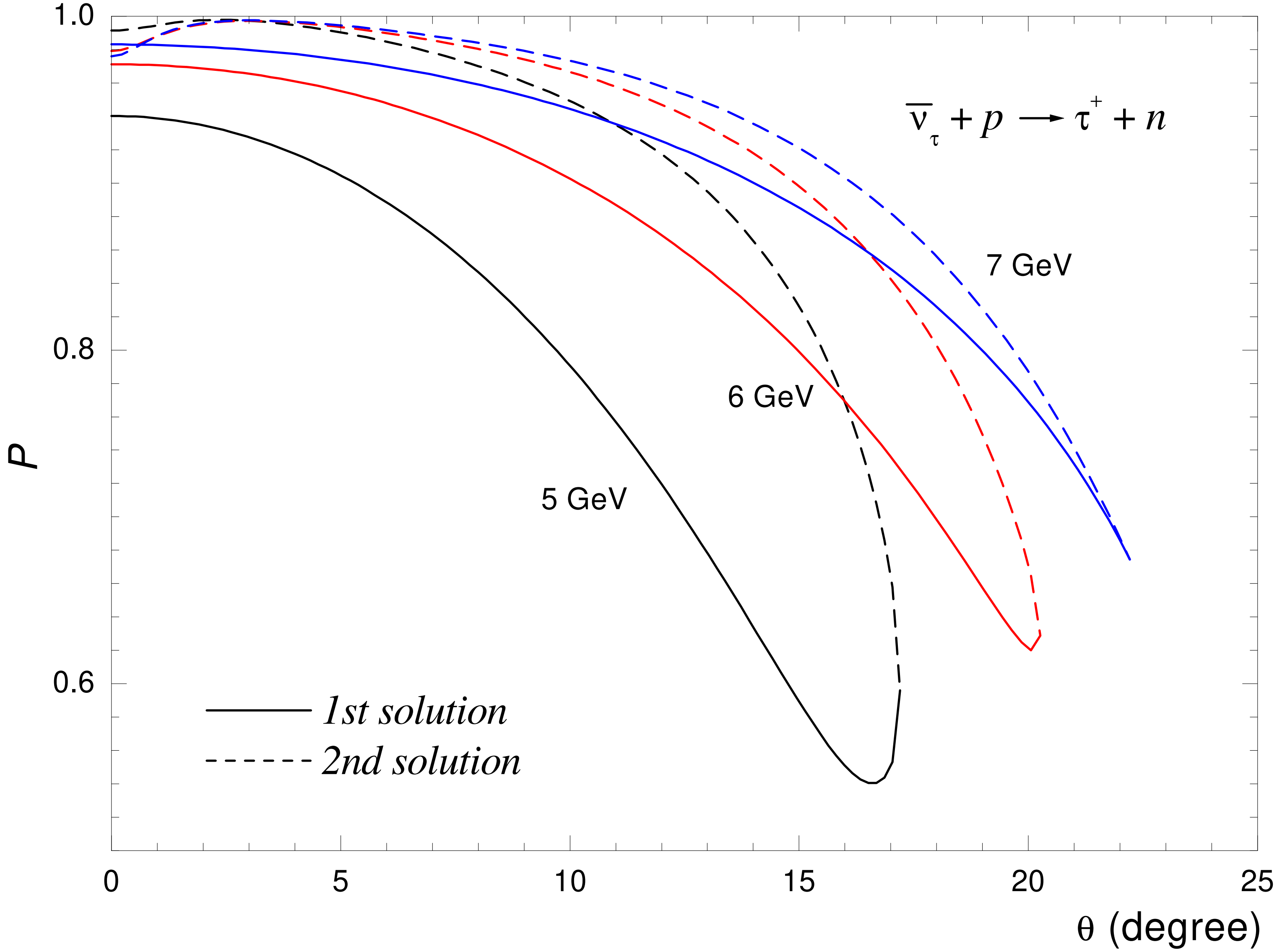
We investigated two models for weak form factors, the standard dipole model and the model by Gari and Krüempelmann (GK) [Phys. Lett. B **274** (1992) 159] improved recently by Lomon [Phys. Rev. C **66** (2002) 045501 (nucl-th/0203081)].

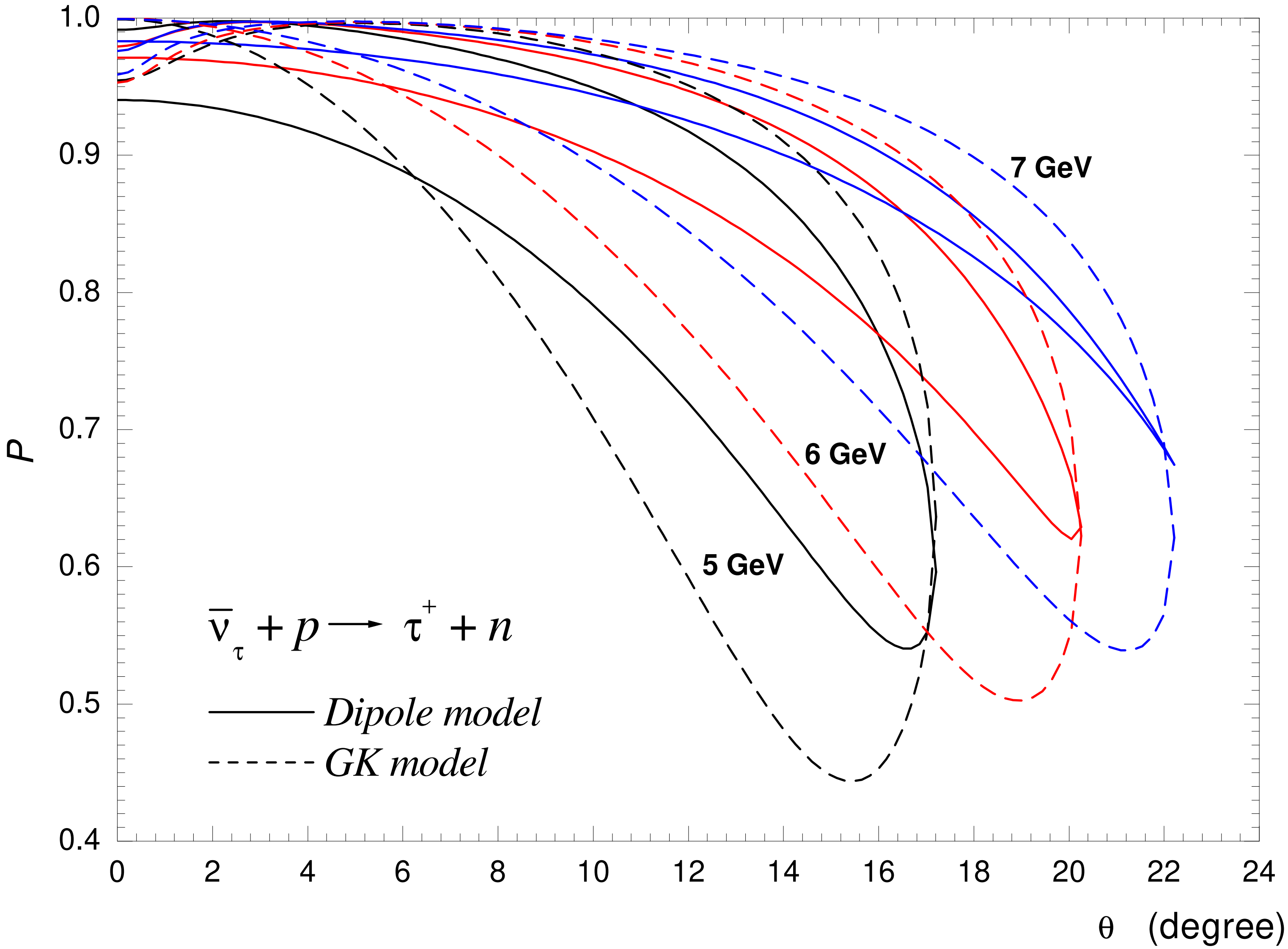


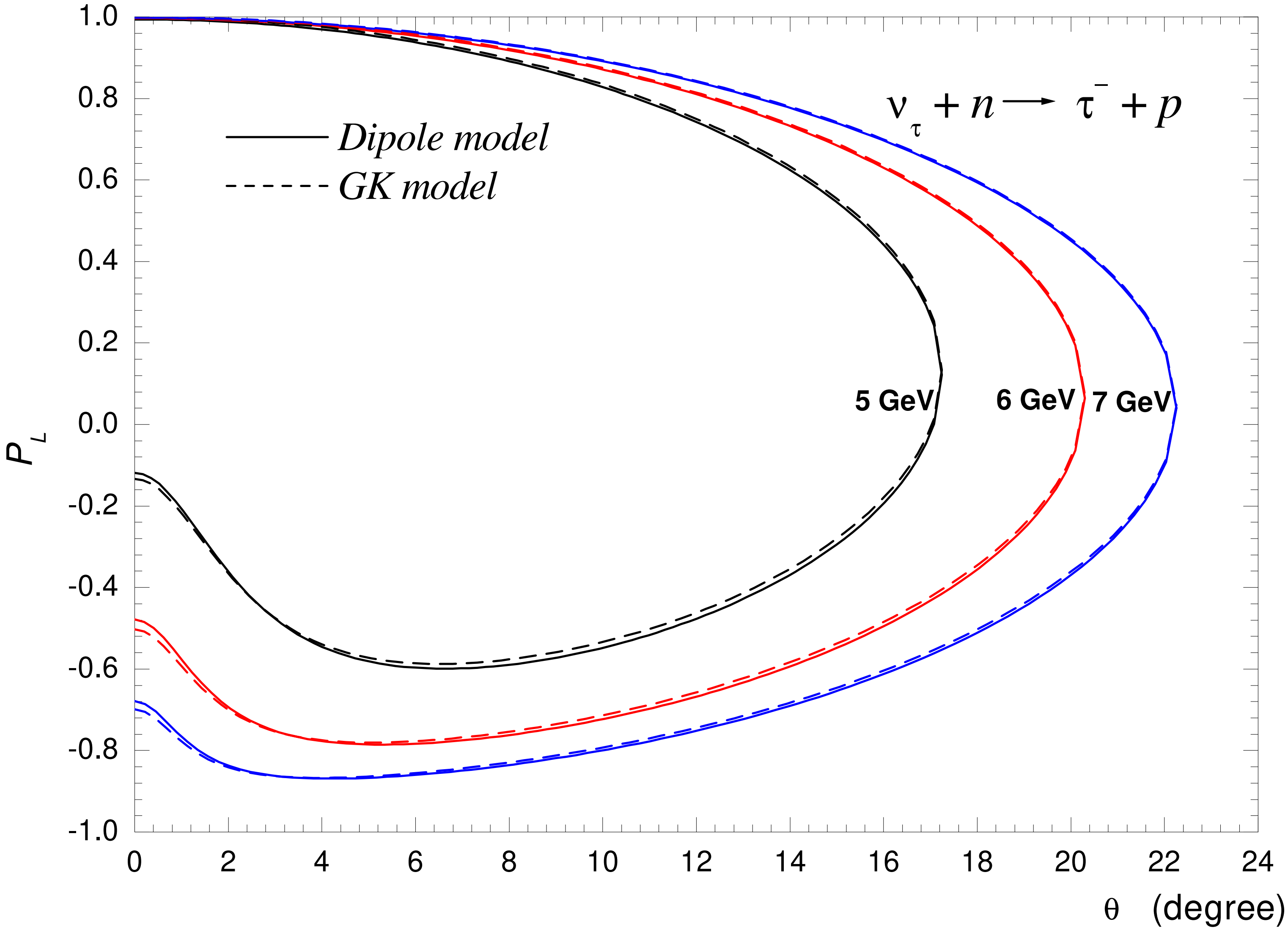
σ (10^-38 cm^2)
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
10^-1
1
10
E_ν (GeV)
ANL 77
BNL 81
Serpukhov 85
FermiLab 83
GGM 77
GGM 79
SKAT 90
ν_μ + n → μ^- + p



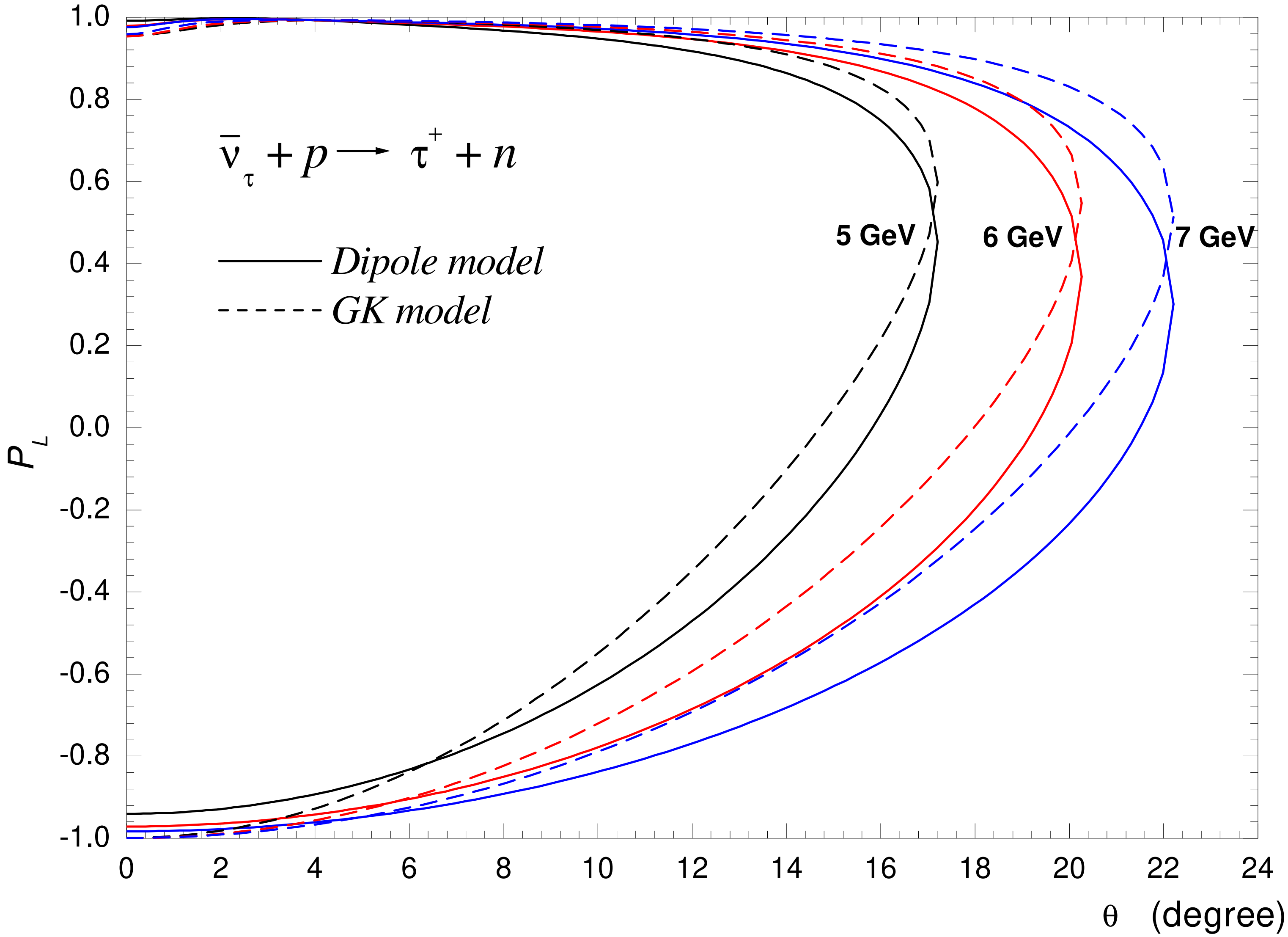
1.00
0.99
0.98
0.97
0.96
0.95
0.94
P
7 GeV
6 GeV
5 GeV
$\nu_\tau + n \longrightarrow \tau^- + p$
Dipole model
GK model
0
2
4
6
8
10
12
14
16
18
20
22
24
θ (degree)


1.0
0.8
0.6
P
0
5
10
15
20
25
θ (degree)
$\overline{\nu}_\tau + p \longrightarrow \tau^+ + n$
7 GeV
6 GeV
5 GeV
1st solution
2nd solution


1.0
0.9
0.8
0.7
0.6
0.5
0.4
P
0
2
4
6
8
10
12
14
16
18
20
22
24
θ (degree)
7 GeV
6 GeV
5 GeV
$\bar{\nu}_\tau + p \longrightarrow \tau^+ + n$
Dipole model
GK model



$\nu_\tau + n \longrightarrow \tau^- + p$
Dipole model
GK model
5 GeV
6 GeV
7 GeV
P_L
θ (degree)
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
0
2
4
6
8
10
12
14
16
18
20
22
24



$\bar{\nu}_\tau + p \longrightarrow \tau^+ + n$
Dipole model
GK model
5 GeV
6 GeV
7 GeV
P_L
θ (degree)
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
0
2
4
6
8
10
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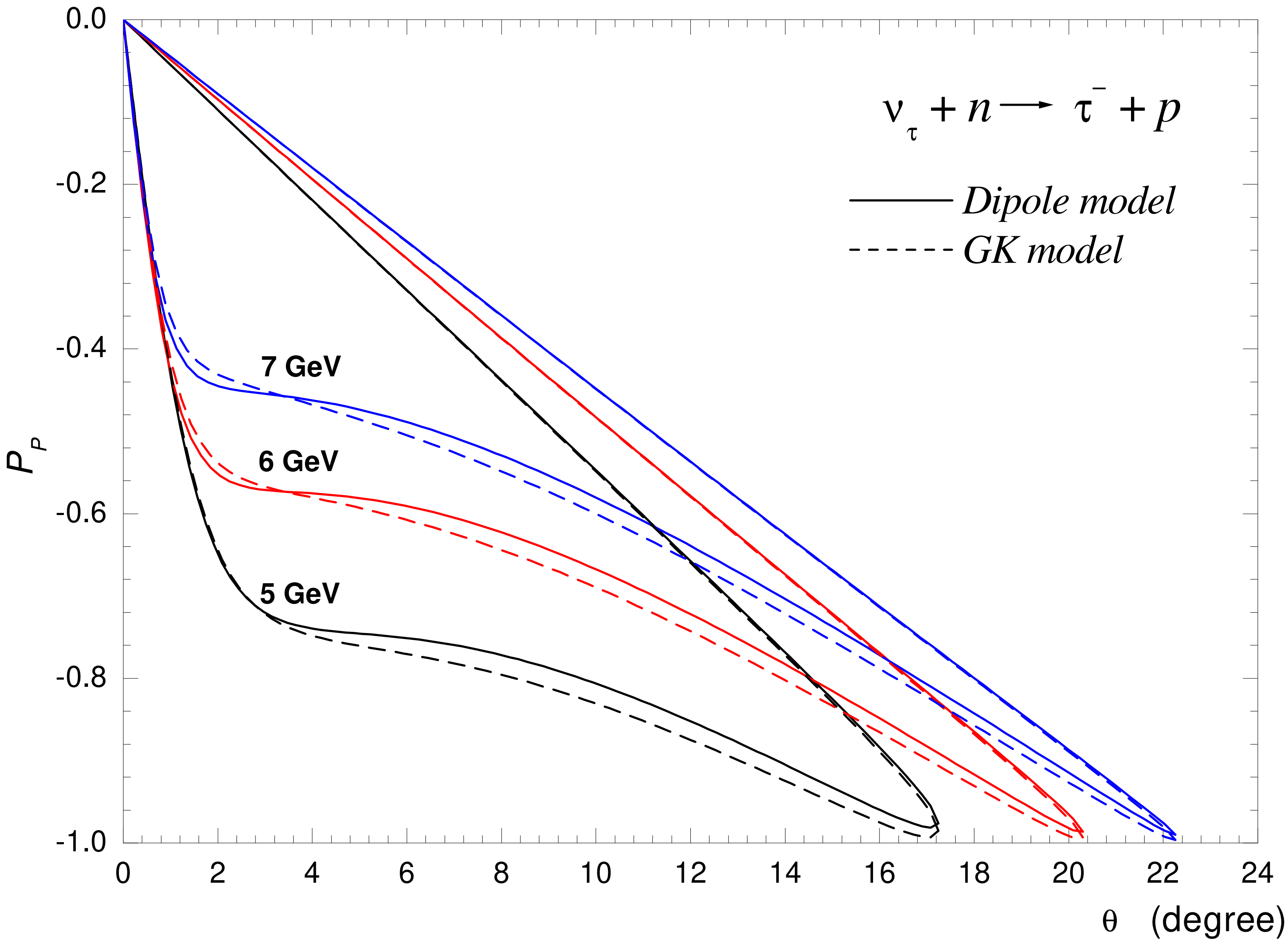
18
20
22
24


$\nu_\tau + n \longrightarrow \tau^- + p$
Dipole model
GK model
7 GeV
6 GeV
5 GeV
P_P
θ (degree)
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
0
2
4
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22
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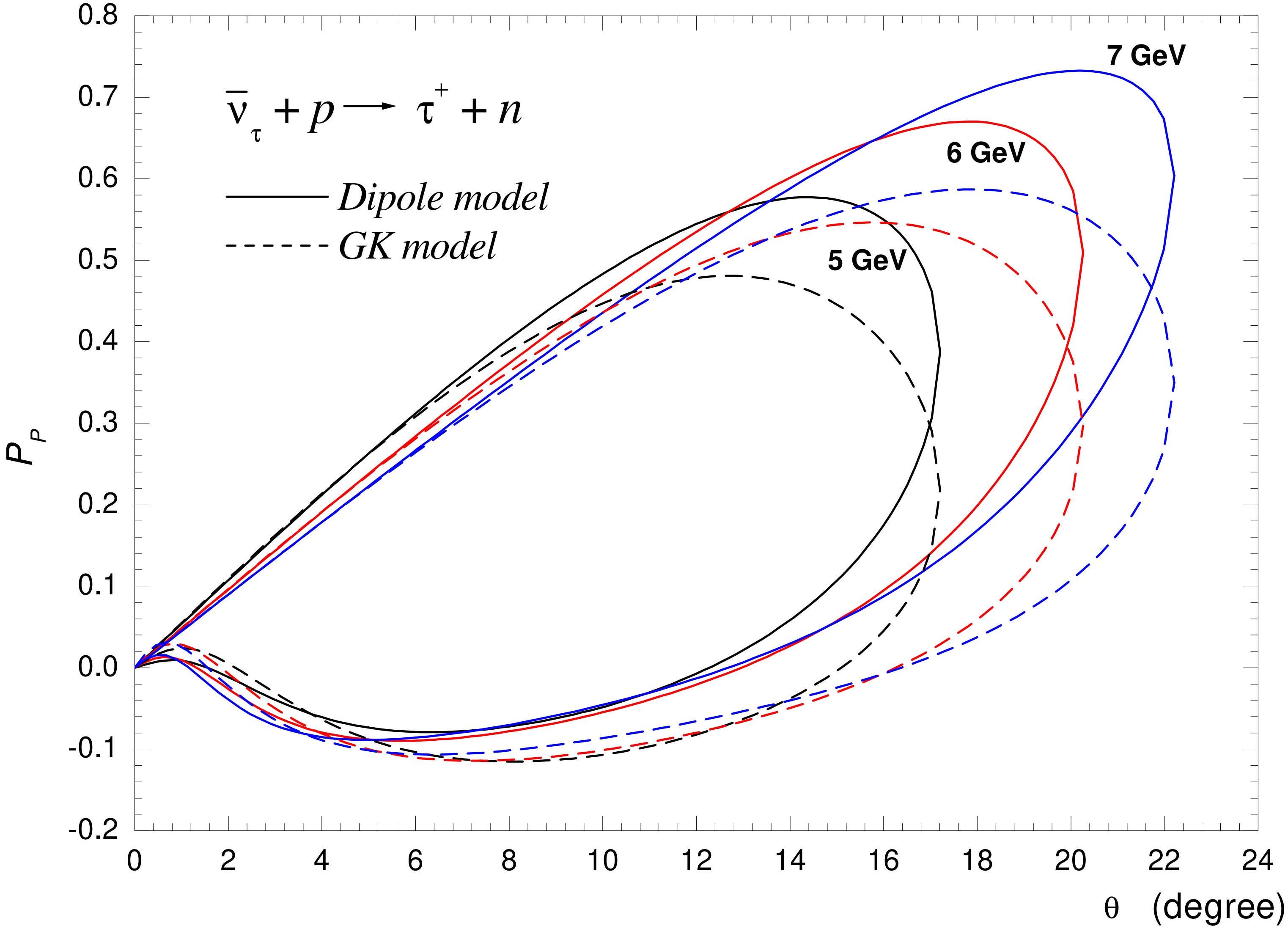


$\bar{\nu}_\tau + p \longrightarrow \tau^+ + n$
Dipole model
GK model
7 GeV
6 GeV
5 GeV
P_P
θ (degree)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
0
2
4
6
8
10
12
14
16
18
20
22
24

Single resonance production

Let us consider first the simplest case of single Δ resonance production,[a]

$$\nu_\ell + n(p) \to \ell^- + \Delta^+ \left(\Delta^{++}\right) , \quad \overline{\nu}_\ell + n(p) \to \ell^+ + \Delta^- \left(\Delta^0\right) .$$

Assuming the isospin symmetry and applying the Wigner-Eckart theorem, the hadronic weak current matrix elements are given by

$$\langle \Delta^+; p' | \widehat{J}_\alpha | n; p \rangle = \langle \Delta^0; p' | \widehat{J}_\alpha | n; p \rangle = \overline{\psi}^\beta \left(p'\right) \Gamma_{\alpha\beta}\, u(p) ,$$
$$\langle \Delta^{++}; p' | \widehat{J}_\alpha | n; p \rangle = \langle \Delta^-; p' | \widehat{J}_\alpha | n; p \rangle = \sqrt{3}\, \overline{\psi}^\beta \left(p'\right) \Gamma_{\alpha\beta}\, u(p) .$$

Here $\psi^\alpha \left(p'\right)$ is the Rarita-Schwinger spin-vector for Δ, $u(p)$ is the Dirac spinor for n, and the vertex tensor is expressed in terms of the 8 weak form factors $C_j^{V,A}$ as

$$\Gamma_{\alpha\beta} = \left[C_3^V \frac{g_{\alpha\beta}\widehat{q} - \gamma_\alpha q_\beta}{M} + C_4^V \frac{g_{\alpha\beta}\left(qp'\right) - p'_\alpha q_\beta}{M^2} + C_5^V \frac{g_{\alpha\beta}\left(qp\right) - p_\alpha q_\beta}{M^2} + C_6^V \frac{q_\alpha q_\beta}{M^2} \right] \gamma_5$$
$$+ C_3^A \frac{g_{\alpha\beta}\widehat{q} - \gamma_\alpha q_\beta}{M} + C_4^A \frac{g_{\alpha\beta}\left(qp'\right) - p'_\alpha q_\beta}{M^2} + C_5^A g_{\alpha\beta} + C_6^A \frac{q_\alpha q_\beta}{M^2} .$$

[a] This example has been investigated recently by K. Hagiwara, K. Mawatari and H. Yokoya [Nucl. Phys. B **668** (2003) 364 (hep-ph/0305324)].

After long and boring calculation we arrive at

$$W_i^{(\mathrm{RES})} = \kappa \cos^2\theta_C M M_\Delta \mathfrak{D} \sum_{j,k=3\text{ to }6} \left(V_i^{jk} C_j^V C_k^V + A_i^{jk} C_j^A C_k^A \right) \quad \text{for } i = 1, 2, 4, 5,$$

$$W_3^{(\mathrm{RES})} = 2\kappa \cos^2\theta_C M M_\Delta \mathfrak{D} \sum_{j,k=3\text{ to }6} K_i^{jk} C_j^V C_k^A \quad \text{and} \quad W_6^{(\mathrm{RES})} = 0.$$

Here θ_C is the Cabibbo mixing angle; $\kappa = 2/3$ for Δ^+/Δ^0 production or 2 for Δ^{++}/Δ^- production;

$$\mathfrak{D} = \frac{1}{\pi} \frac{W\Gamma(W)}{\left(W^2 - M_\Delta^2\right)^2 + W^2\Gamma^2(W)} \quad \text{and} \quad \Gamma(W) = \Gamma_\Delta \sqrt{\frac{\lambda\left(W^2, M^2, m_\pi\right)}{\lambda\left(M_\Delta^2, M^2, m_\pi\right)}}$$

are the Breit-Wigner factor and the running widths of Δ,[a] respectively. The coefficients V_i^{jk}, A_i^{jk} and K_i^{jk} are found to be cubic polynomials over the invariant variables

$$w = \frac{(pq)}{M^2} = \frac{E_\nu - E_\ell}{M} \quad \text{and} \quad x = \frac{-q^2}{2(pq)} = \frac{2E_\nu\left(E_l - P_l\cos\theta\right) - m^2}{2M\left(E_\nu - E_l\right)}$$

and over the parameter $\zeta = M/M_\Delta$. Only *70* among the *144* coefficients are nonzero.

[a] $\Gamma(W)$ is estimated by the S-wave $\Delta \to N\pi$ decay; $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2(ab + bc + ca)$.

Resonance production in the Rein–Sehgal model [a]

We consider the CC reaction $\nu_\ell N \to \ell N'\pi$. Let us write the general expression for the elements of the polarization density matrix:

$$\frac{d\sigma_{\lambda\lambda'}}{dq^2 dW} = \frac{G_F^2 \cos^2\theta_C}{4\pi^2}\left(\frac{-q^2}{\mathcal{Q}^2}\right)\frac{W}{M}\sum_{i=L,R,S} c_i^\lambda c_i^{\lambda'} \sigma_i^{\lambda\lambda'}\left(q^2, W\right),$$

where

$$W^2 = p'^2 = (p+q)^2 \quad \text{and} \quad \mathcal{Q} = |\mathbf{q}| = \frac{1}{2M}\sqrt{\left(W^2 - M^2 - q^2\right)^2 - 4M^2q^2}.$$

In the resonance rest frame (RRF), the leptonic current, j_λ^β, can be written in terms of 3 polarization vectors:

$$j_\lambda^\beta = C^{-1}\left[c_S^\lambda e_{(\lambda)}^\beta + c_R^\lambda e_R^\beta + c_L^\lambda e_L^\beta\right], \quad C = \frac{\mathcal{Q}}{2E_\nu\sqrt{-2q^2}},$$

$$e_{(\lambda)}^\beta = \frac{1}{\sqrt{-q^2}}\left(\mathcal{Q}_{(\lambda)}^\star, 0, 0, \nu_{(\lambda)}^\star\right), \quad e_L^\beta = \frac{1}{\sqrt{2}}(0,1,-i,0), \quad e_R^\beta = \frac{1}{\sqrt{2}}(0,-1,-i,0).$$

[a] D. Rein and L. Sehgal, Ann. Phys. **133** (1981) 79.

It is convenient to denote

$$j_\lambda^{0,1,3} = \mathcal{J}_\lambda^{0,1,3}, \qquad j_\lambda^2 = -i\,\mathcal{J}_\lambda^2.$$

The components $\mathcal{J}_\lambda^\alpha$ are real and moreover one can force the $\mathcal{J}_\lambda^0$ to be positive.

Next we introduce the 4-vector

$$r = r(\lambda) = k' - ms \stackrel{\text{def}}{=} \varrho_1(\lambda)k' + \varrho_2(\lambda)p.$$

Here s is the usual axial 4-vector of the lepton spin whose components are

$$s_0 = \frac{(\boldsymbol{\xi}\mathbf{k}')}{m}, \quad \mathbf{s} = \boldsymbol{\xi} + \frac{(\boldsymbol{\xi}\mathbf{k}')\,\mathbf{k}'}{m\,(E_\ell + m)},$$

where $\boldsymbol{\xi} = \mathbf{P}/P$ is the unit vector directed along the spin in the lepton rest frame.

The functions $\varrho_{1,2}$ are found to be

$$\varrho_1 = 1 + \lambda\frac{E_\ell}{P_\ell} \quad \text{and} \quad \varrho_2 = -\lambda\frac{m^2}{MP_\ell}.$$

The neutrino and lepton energies in RRF are

$$E_\nu^\star = \frac{1}{2W}\left[2ME_\nu + q^2 - m^2\right] \equiv \varepsilon_\nu \quad \text{and} \quad E_\ell^\star = \frac{1}{2W}\left[2ME_\ell - q^2 - m^2\right] \equiv \varepsilon_\ell.$$

Then the components of the vector r in RRF can be written as[a]

$$r_0^\star = \varrho_1 \varepsilon_\ell + \varrho_2 \left[(\mathcal{Q}^\star)^2 + M^2 \right]^{1/2}, \quad r_1^\star = \varrho_1 k_1^\star, \quad r_2^\star = 0, \quad r_3^\star = \varrho_1 {k'_3}^\star + \varrho_2 \mathcal{Q}^\star.$$

The explicit form of the diagonal over helicities part of the leptonic tensor

$$j_\alpha^\lambda j_\beta^{\lambda *} = L_{\alpha\beta}(k, r(\lambda)) \mp L^5_{\alpha\beta}(k, r(\lambda)), \quad \text{where}$$

$$L_{\alpha\beta}(k, r) = \mathrm{Tr}\left(\hat{r} \gamma_\alpha \hat{k} \gamma_\beta \right) = 4 \left[k_\alpha r_\beta + r_\alpha k_\beta - g_{\alpha\beta}(kr) \right],$$

$$L^5_{\alpha\beta}(k, r) = \mathrm{Tr}\left(\hat{r} \gamma_\alpha \hat{k} \gamma_\beta \gamma_5 \right) = 4 i \epsilon_{\alpha\beta\gamma\delta} k^\gamma r^\delta,$$

provides the equations for obtaining the current $\mathcal{J}_\alpha^\lambda$:

$$\mathcal{J}_0^\lambda \mathcal{J}_0^\lambda = \ 4 \left[2\varepsilon_\nu r_0^\star - (kr) \right], \quad \mathcal{J}_0^\lambda \mathcal{J}_1^\lambda = 4 \left(\varepsilon_\nu r_1^\star + r_0^\star k_1^\star \right),$$

$$\mathcal{J}_0^\lambda \mathcal{J}_2^\lambda = \pm 4 \left(k_1^\star r_3^\star - k_3^\star r_1^\star \right), \quad \mathcal{J}_0^\lambda \mathcal{J}_3^\lambda = 4 \left(\varepsilon_\nu r_3^\star + r_0^\star k_3^\star \right).$$

Now we ready to find out all the rest ingredients necessary for calculating the spin density matrix.

---

[a] In the laboratory frame, $r = (\lambda - E_\ell / P_\ell)(P_\ell, \lambda \mathbf{k}')$. The inner product, (kr), is given by $(kr) = \varrho_1 \left(m^2 - q^2 \right)/2 + \varrho_2 M E_\nu$.

These are

$$c_L^\lambda = \frac{C}{\sqrt{2}}\left(\mathcal{J}_1^\lambda + \mathcal{J}_2^\lambda\right), \quad c_R^\lambda = \frac{C}{\sqrt{2}}\left(\mathcal{J}_2^\lambda - \mathcal{J}_1^\lambda\right), \quad c_S^\lambda = C\sqrt{|(\mathcal{J}_0^\lambda)^2 - (\mathcal{J}_3^\lambda)^2|},$$

$$\mathcal{Q}_{(\lambda)}^\star = \frac{C\sqrt{-q^2}}{c_S^\lambda}\mathcal{J}_0^\lambda, \quad \nu_{(\lambda)}^\star = \frac{C\sqrt{-q^2}}{c_S^\lambda}\mathcal{J}_3^\lambda.$$

Isospin structure of the amplitudes

The L, R and S contribution to the full cross section are written in terms of the transition amplitudes $A_\varkappa^\lambda$:

$$\sigma_{L,R}^{\lambda\lambda'} = \frac{\pi W}{2M}\left(A_{\pm3}^\lambda A_{\pm3}^{\lambda'} + A_{\pm1}^\lambda A_{\pm1}^{\lambda'}\right),$$

$$\sigma_S^{\lambda\lambda'} = \frac{\pi W}{2M}\left(\frac{\mathcal{Q}^2}{-q^2}\right)\frac{M^2}{W^2}\left(A_{0+}^\lambda A_{0+}^{\lambda'} + A_{0-}^\lambda A_{0-}^{\lambda'}\right),$$

with $\varkappa = \pm3,\ \pm1,\ 0\pm$. In the following, we use the definitions (and similar notation) for the amplitudes described by Rein and Sehgal:

$$A_{\varkappa}^{\lambda}\left(p\pi^{+}\right) = \sqrt{3}\sum_{(I=3/2)} a_{\varkappa}^{\lambda}\left(\mathcal{N}_{3}^{*+}\right),$$

$$\sqrt{3}A_{\varkappa}^{\lambda}\left(p\pi^{0}\right) = \sqrt{2}\sum_{(I=3/2)} a_{\varkappa}^{\lambda}\left(\mathcal{N}_{3}^{*+}\right) - \sum_{(I=1/2)} a_{\varkappa}^{\lambda}\left(\mathcal{N}_{1}^{*+}\right),$$

$$\sqrt{3}A_{\varkappa}^{\lambda}\left(n\pi^{+}\right) = \sum_{(I=3/2)} a_{\varkappa}^{\lambda}\left(\mathcal{N}_{3}^{*+}\right) + \sqrt{2}\sum_{(I=1/2)} a_{\varkappa}^{\lambda}\left(\mathcal{N}_{1}^{*+}\right).$$

Here we have to stress only those resonances are allowed to interfere which have the same spin and orbital angular momentum. Let us show a typical example:

$$\begin{aligned}
3A_{(\varkappa)}^{\lambda}A_{(\varkappa)}^{\lambda'}(n\pi^{+}) = {} & \left[\sum a_{\varkappa}^{\lambda}(S_{31}^{+}) + \sum a_{\varkappa}^{\lambda}(S_{11}^{+})\right]\left[\sum a_{\varkappa}^{\lambda'}(S_{31}^{+}) + \sum a_{\varkappa}^{\lambda'}(S_{11}^{+})\right] \\
& + \sum_{j=1,3}\left[\sum a_{\varkappa}^{\lambda}(P_{3j}^{+}) + \sum a_{\varkappa}^{\lambda}(P_{1j}^{+})\right]\left[\sum a_{\varkappa}^{\lambda'}(P_{3j}^{+}) + \sum a_{\varkappa}^{\lambda'}(P_{1j}^{+})\right] \\
& + \sum_{j=3,5}\left[\sum a_{\varkappa}^{\lambda}(D_{3j}^{+}) + \sum a_{\varkappa}^{\lambda}(D_{1j}^{+})\right]\left[\sum a_{\varkappa}^{\lambda'}(P_{3j}^{+}) + \sum a_{\varkappa}^{\lambda'}(P_{1j}^{+})\right] \\
& + \sum_{j=5,7}\left[\sum a_{\varkappa}^{\lambda}(F_{3j}^{+}) + \sum a_{\varkappa}^{\lambda}(F_{1j}^{+})\right]\left[\sum a_{\varkappa}^{\lambda'}(P_{3j}^{+}) + \sum a_{\varkappa}^{\lambda'}(P_{1j}^{+})\right].
\end{aligned}$$

Every amplitude $a_{\varkappa}^{\lambda}\left(\mathcal{N}_{\imath}^{*+}\right)$ referring to a single resonance consists of two factors which describe the production of the resonance $\mathcal{N}_{\imath}^{*+}$ and its subsequent decay into the $\mathcal{N}\pi$ final state:

$$a_{\varkappa}^{\lambda}\left(\mathcal{N}_{\imath}^{*}\right) = f_{\varkappa}^{\lambda}\left(\nu\mathcal{N} \to \mathcal{N}_{\imath}^{*}\right)\eta(\mathcal{N}_{\imath}^{*} \to \mathcal{N}\pi) \equiv f_{\varkappa}^{\lambda(\imath)}\,\eta^{(\imath)}.$$

The resonance production amplitudes, $f_{\varkappa}^{\lambda(\imath)}$, are collected in Table II of the RS paper.

The corresponding decay amplitudes, $\eta^{(\imath)}$, can be split into the three factors:

$$\eta^{(\imath)} = \eta^{(\imath)}_{BW}(W)\,\sqrt{\chi_\imath}\,\mathrm{sign}(\mathcal{N}^*_\imath),$$

where

$$\eta^{(\imath)}_{BW}(W) = \sqrt{\frac{1}{2\pi N_\imath}\left[\frac{\Gamma_\imath}{(W-M_\imath)^2+\Gamma_\imath^2/4}\right]},$$

$$N_\imath = \frac{1}{2\pi}\int_{W_{min}}^{\infty} dW\left[\frac{\Gamma_\imath}{(W-M_\imath)^2+\Gamma_\imath^2/4}\right],$$

$$\Gamma_\imath = \Gamma^0_\imath\left[\frac{\lambda(W^2,M^2,m_\pi)}{\lambda(M_\imath^2,M^2,m_\pi)}\right]^{L+1/2},$$

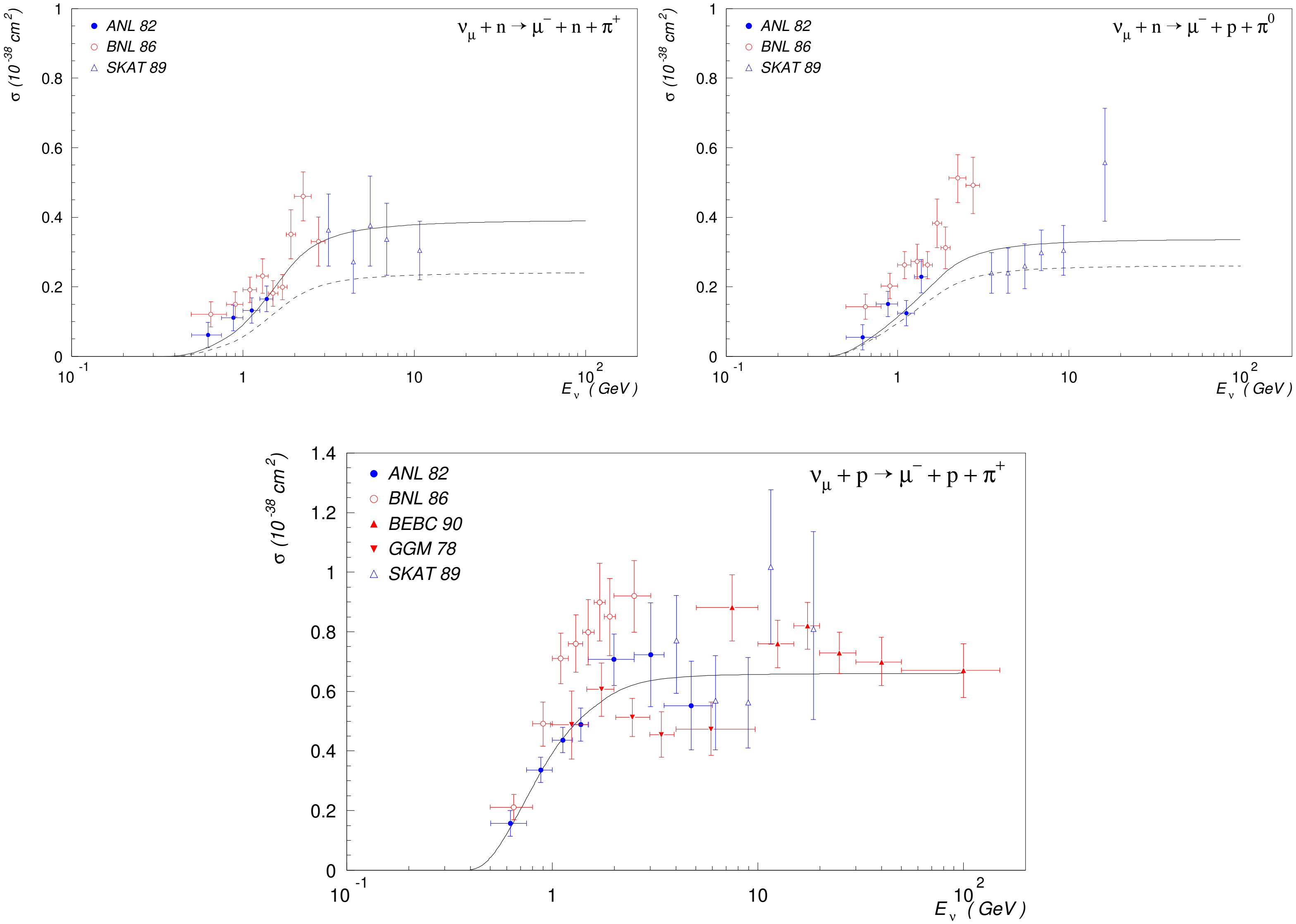
$\chi_\imath$ is the branching ratio for the decay of the resonance into the $\mathcal{N}\pi$ final state,[a] sign $(\mathcal{N}^*_\imath)$ is the pure sign given in Table III of the RS paper.

[a] In our numerical calculations, these are updated according to the recent PDG data.

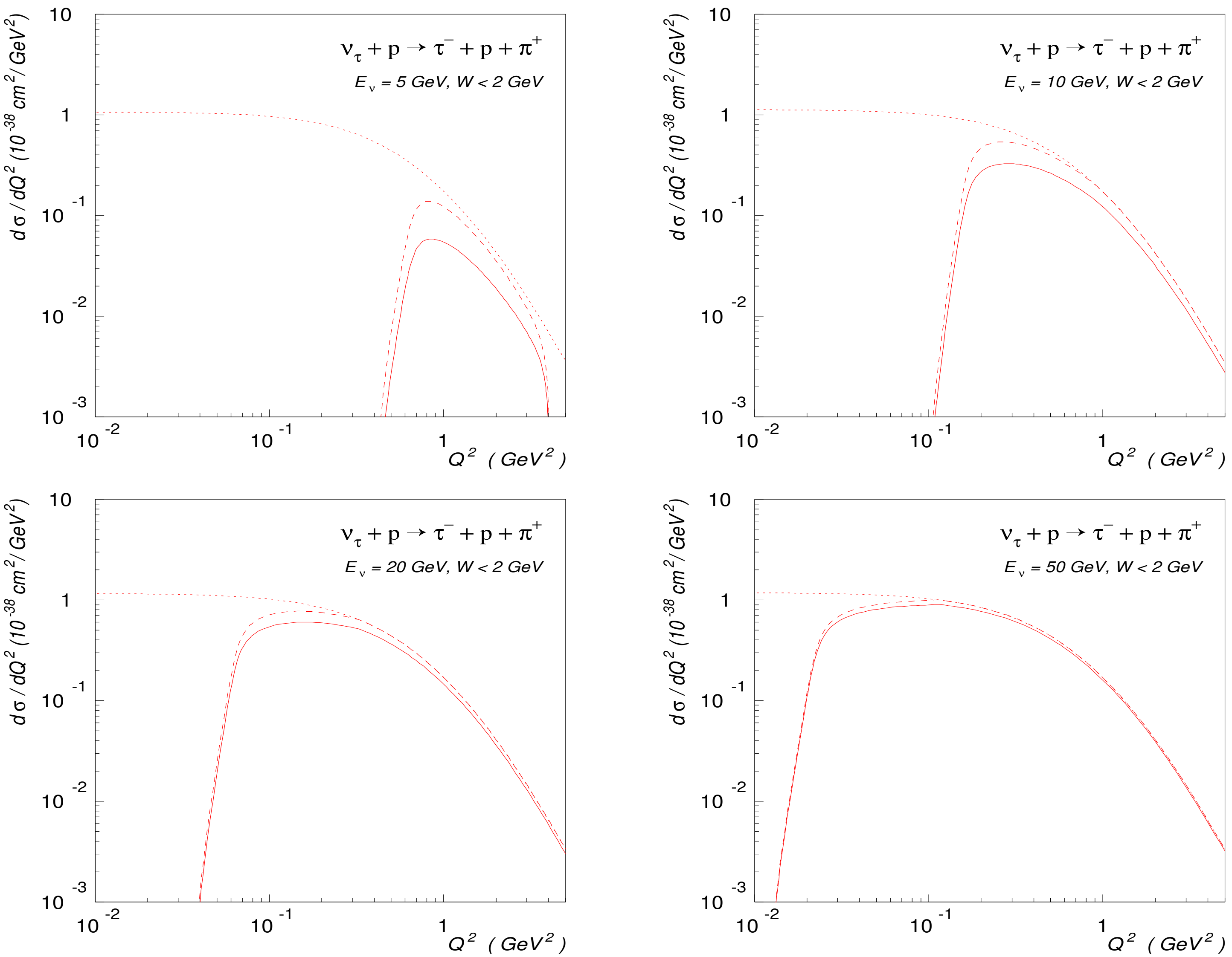
Note:

In comparison with the original paper, the structure of the vector $e^{\alpha}_{(\lambda)}$ has been changed by including the spin dependence. So we have to calculate the inner products $J^{V,A}_{\alpha} \cdot e^{\mu}_{(\lambda)}$, where $J^{V,A}_{\alpha}$ are the vector and axial hadronic currents in the FKR model. The new definitions for the structures S^V, B^A and C^A involved into the model are the following:

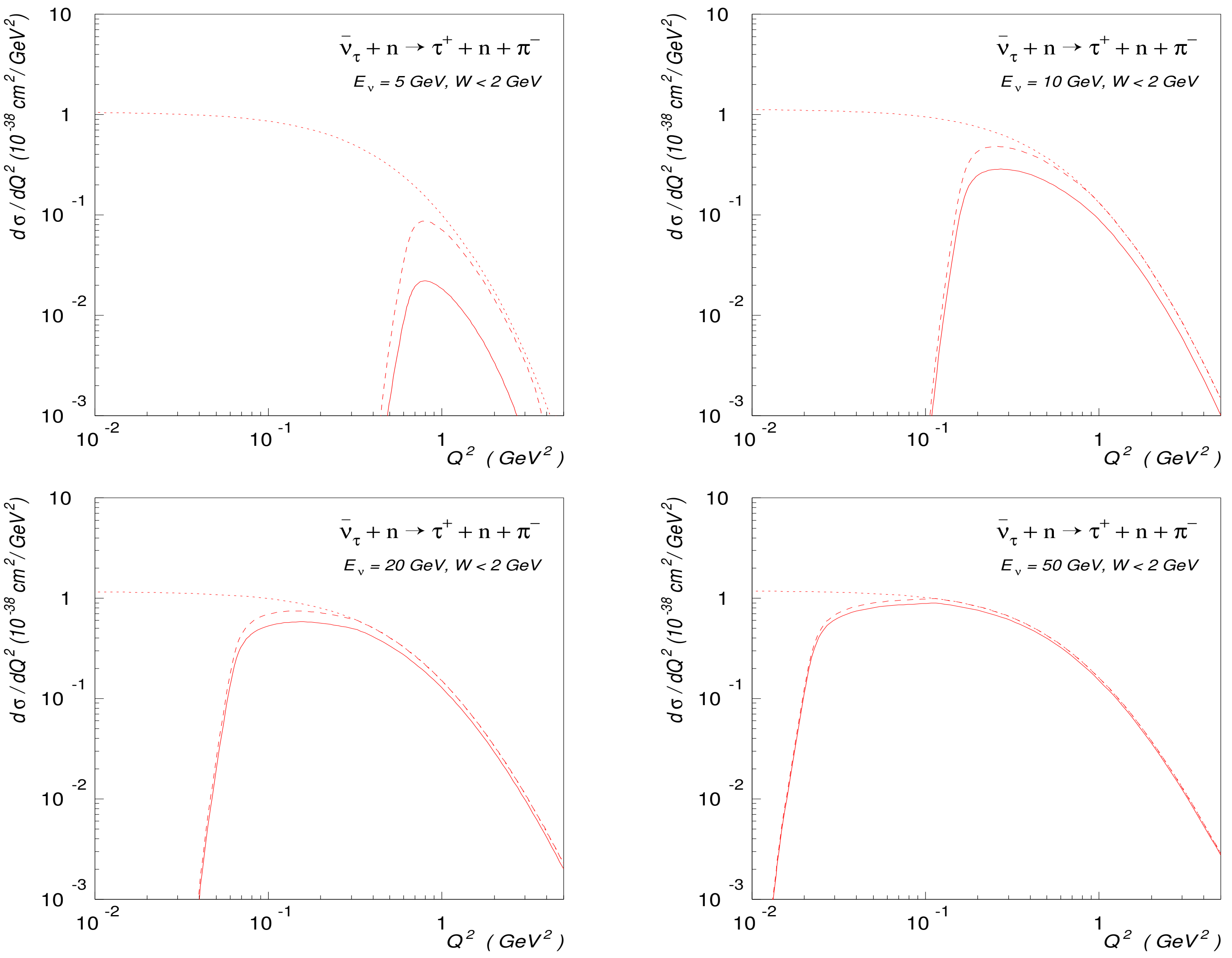
$$
\begin{aligned}
S^V &= \left(\nu^{\star}_{(\lambda)}\nu^{\star} - \mathcal{Q}^{\star}_{(\lambda)}\mathcal{Q}^{\star}\right)\left(1 - \frac{q^2}{M^2} - \frac{3W}{M}\right)\frac{G^V\left(q^2\right)}{6\mathcal{Q}^2}, \\
B^A &= \sqrt{\frac{\Omega}{2}}\left(\mathcal{Q}^{\star}_{(\lambda)} + \nu^{\star}_{(\lambda)}\frac{\mathcal{Q}^{\star}}{2mg^2}\right)\frac{ZG^A\left(q^2\right)}{3W\mathcal{Q}^{\star}}, \\
C^A &= \left[\left(\mathcal{Q}^{\star}_{(\lambda)}\mathcal{Q}^{\star} - \nu^{\star}_{(\lambda)}\nu^{\star}\right)\left(\frac{1}{3} + \frac{\nu^{\star}}{2mg^2}\right)\right. \\
&\qquad \left. + \nu^{\star}_{(\lambda)}\left(\frac{2}{3}W + \frac{q^2}{2mg^2} + \frac{N\Omega}{6mg^2}\right)\right]\frac{ZG^A\left(q^2\right)}{2W\mathcal{Q}^{\star}}.
\end{aligned}
$$

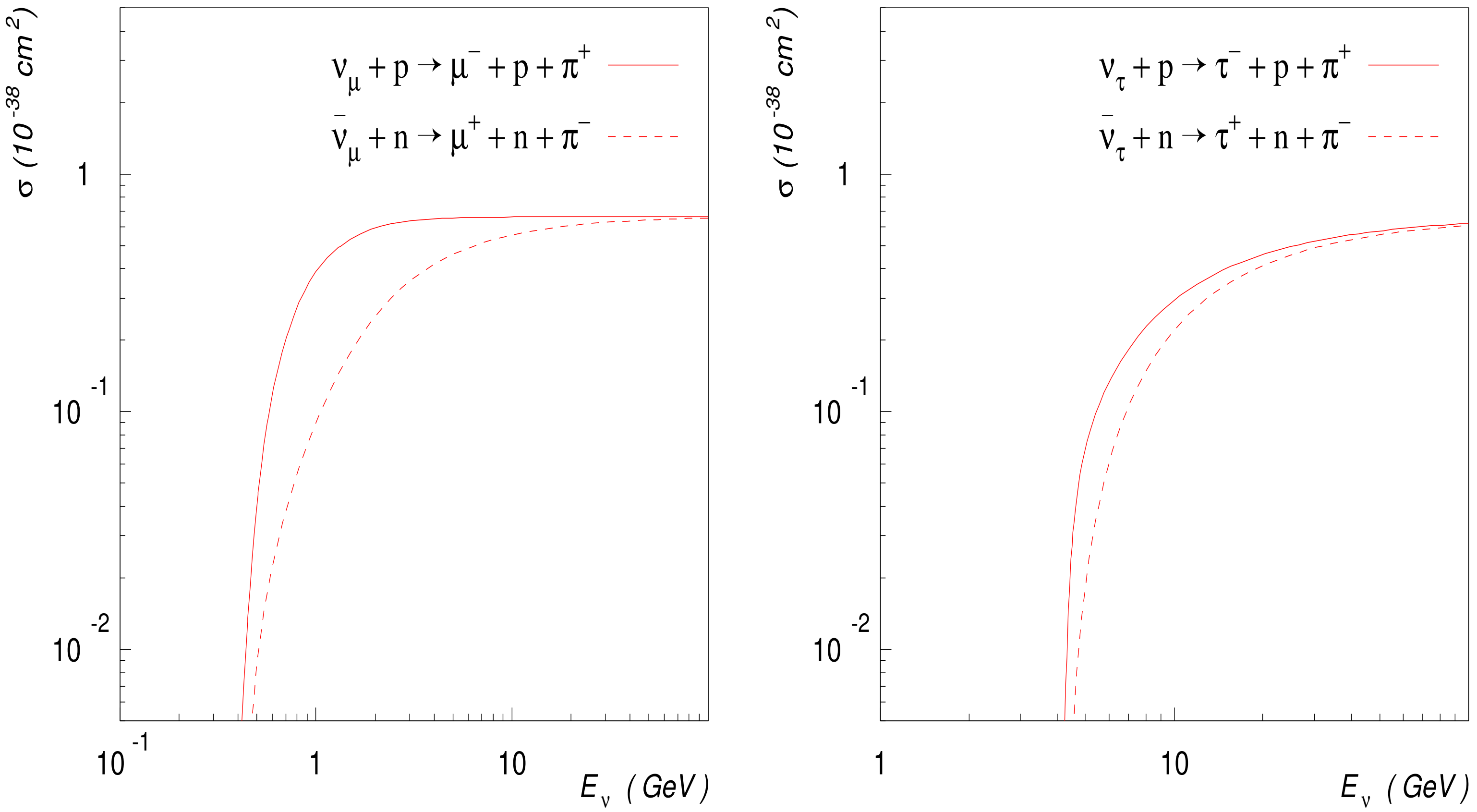


σ (10^-38 cm^2)
ANL 82
BNL 86
SKAT 89
ν_μ + n → μ^- + n + π^+
E_ν (GeV)
ν_μ + n → μ^- + p + π^0
ν_μ + p → μ^- + p + π^+
BEBC 90
GGM 78

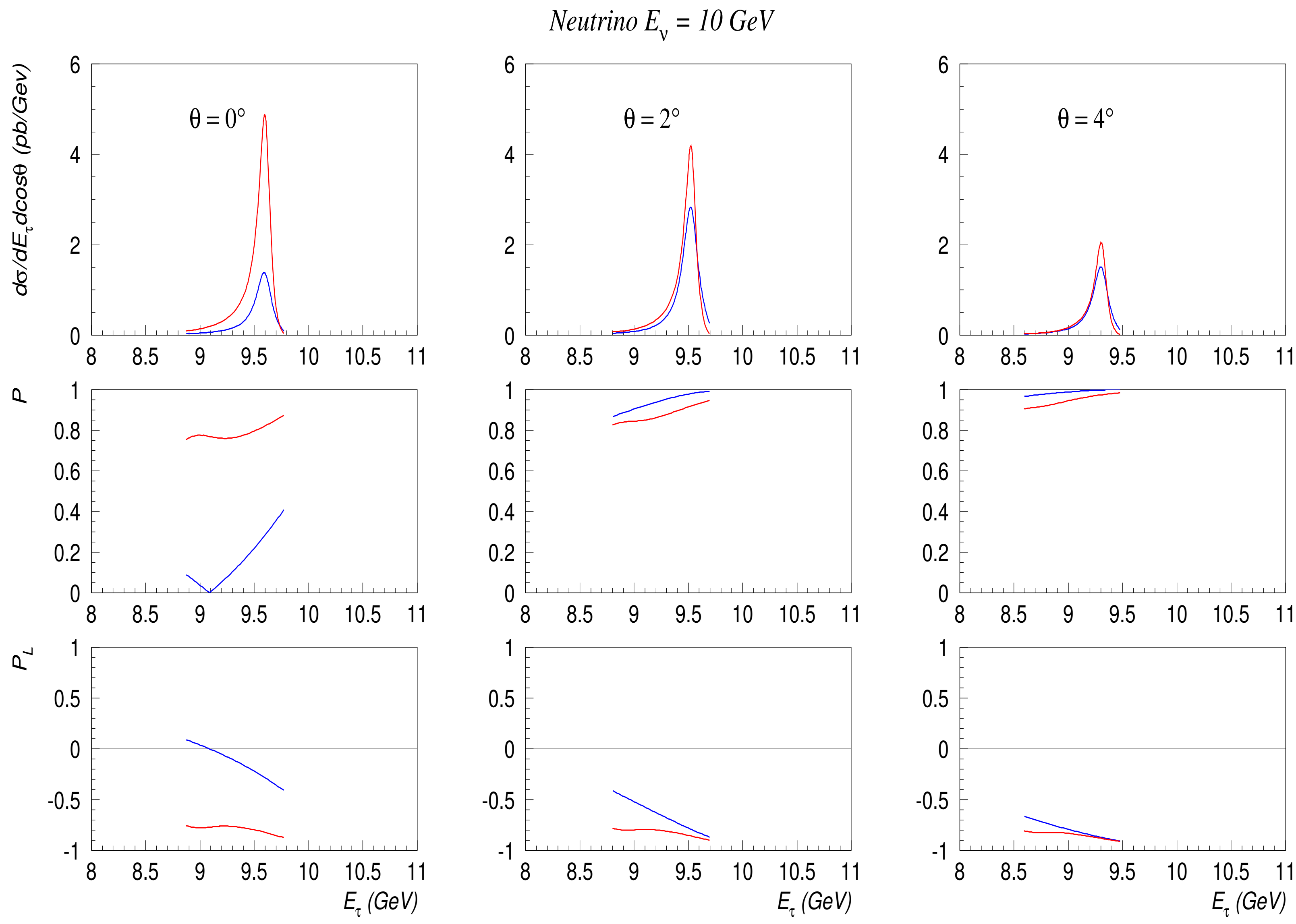


$\nu_\tau + p \to \tau^- + p + \pi^+$
E_ν = 5 GeV, W < 2 GeV
$\nu_\tau + p \to \tau^- + p + \pi^+$
E_ν = 10 GeV, W < 2 GeV
$\nu_\tau + p \to \tau^- + p + \pi^+$
E_ν = 20 GeV, W < 2 GeV
$\nu_\tau + p \to \tau^- + p + \pi^+$
E_ν = 50 GeV, W < 2 GeV
$d\sigma / dQ^2\ (10^{-38}\ cm^2/GeV^2)$
$Q^2\ (GeV^2)$


$\bar{\nu}_\tau + n \to \tau^+ + n + \pi^-$
E_ν = 5 GeV, W < 2 GeV
$\bar{\nu}_\tau + n \to \tau^+ + n + \pi^-$
E_ν = 10 GeV, W < 2 GeV
$\bar{\nu}_\tau + n \to \tau^+ + n + \pi^-$
E_ν = 20 GeV, W < 2 GeV
$\bar{\nu}_\tau + n \to \tau^+ + n + \pi^-$
E_ν = 50 GeV, W < 2 GeV
$d\sigma / dQ^2$ (10^{-38} cm^2/ GeV^2)
Q^2 (GeV^2)
10
1
10^{-1}
10^{-2}
10^{-3}

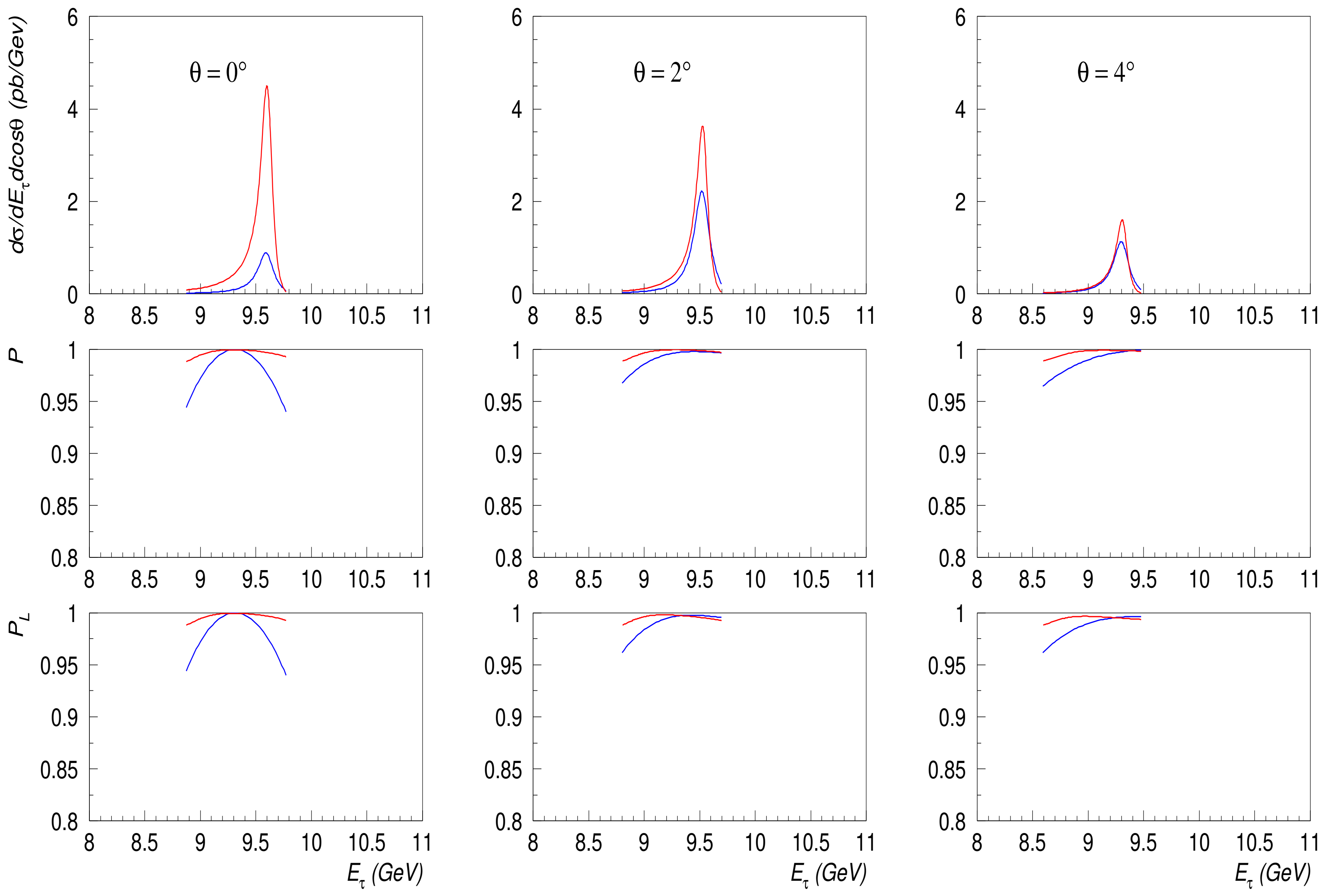


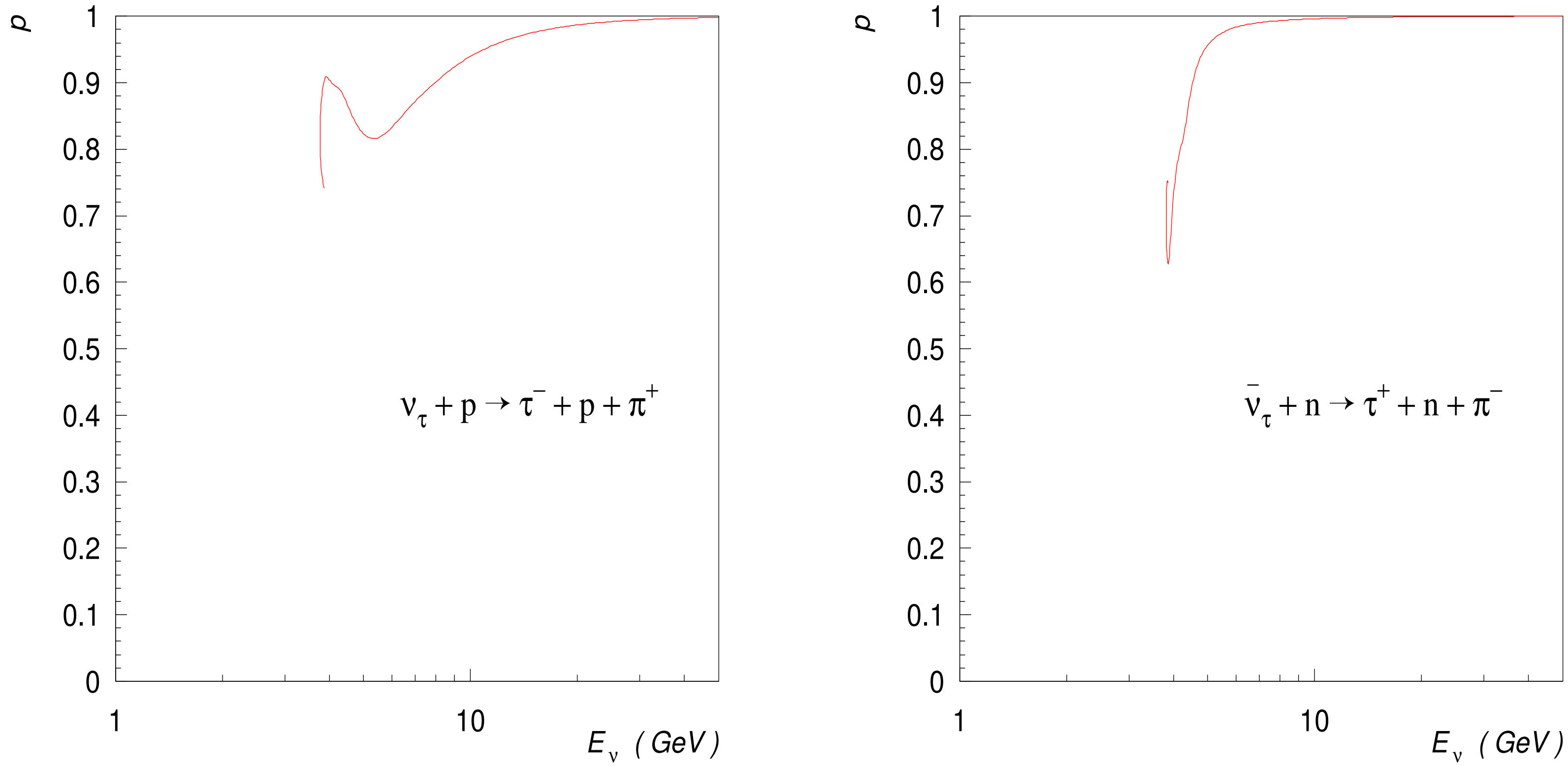
σ (10^-38 cm^2)
1
10^-1
10^-2
10^-1
1
10
E_ν (GeV)
ν_μ + p → μ^- + p + π^+
ν̄_μ + n → μ^+ + n + π^-
1
10
ν_τ + p → τ^- + p + π^+
ν̄_τ + n → τ^+ + n + π^-



Neutrino E_ν = 10 GeV
$d\sigma/dE_\tau dcos\theta$ (pb/Gev)
$\theta = 0°$
$\theta = 2°$
$\theta = 4°$
P
P_L
E_τ (GeV)

Antineutrino E_ν = 10 GeV





The degree of polarization of $\tau^\pm$, $P = \sqrt{P_P^2 + P_L^2}$, averaged over the τ lepton energy and scattering angle.