ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

3/4

E2-96-373
HEP-TH 9610073

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PARK—TARTER MATRIX FOR A DYON-DYON SYSTEM

Submitted to the II International Workshop «Classical and Quantum Integrable Systems», July 8—12, 1996, Dubna, Russia

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1996

1 Introduction

In this paper, we have calculated the matrix between the spherical and parabolic bases of a dyon–dyon system [1] belonging to the same energy level. This matrix is a generalization of the Park–Tarter matrix known from the theory of hydrogen atom [2, 3] to the case when the Coulomb center carries not only the electric but also magnetic charge. Like the Park–Tarter matrix, our matrix is expressed through the Clebsch–Gordan coefficients $C^{c\gamma}_{a\alpha;b\beta}$, however, in our case $a \neq b$, in contrast to the case of a hydrogen atom. We have also traced the connection of the dyon–dyon problem with that of a 4-dimensional isotropic oscillator. As is known [4], these problems are related to each other by the Kustaanheimo–Stiefel transformation [5] supplemented with the 4th (angular) coordinate. We have shown that the coefficients $C^{c\gamma}_{a\alpha;b\beta}$ coincide with the ones [6] of the expansion of the double polar basis over the Euler basis of a 4-dimensional isotropic oscillator.

2 Dyon–Dyon System

A dyon–dyon system in the space $\mathbb{R}^3$ is described by the equation

$$\left[\left(\frac{\partial}{\partial x_j} - \frac{ie}{\hbar c}A_j\right)^2 - \frac{s^2}{r^2}\right]\psi + \frac{2M_0}{\hbar^2}\left(\epsilon^s + \frac{e^2}{r}\right)\psi = 0 \tag{1}$$

where

$$A_j = \frac{g x_3}{r(r^2 - x_3^2)}(-x_2, x_1, 0)$$

and $s = eg/\hbar c = 0, \pm 1/2, \pm 1, \ldots$. Each value of s describes its particular dyon–dyon system. At $s = 0$, eq.(1) is reduced to the Schrödinger equation for a hydrogen atom. When $s \neq 0$, equation (1) preserves O(4)-symmetry and therefore variables in it are separated into spherical, parabolic, and prolate spheroidal coordinates [1].

The system (1) possesses a singularity on the axis x_3. It is also possible to consider systems with singularities either on the semiaxis $x_3 > 0$ or on $x_3 < 0$, i.e. they are described by the vector potentials

$$A_j^{(\pm)} = \frac{g}{r(r \mp x_3)}(\mp x_2, \pm x_1, 0)$$

and are connected with the system (1) by the gauge transformations

$$A_j^{(\pm)} = A_j + \frac{\partial f^{(\pm)}}{\partial x_j}, \qquad \psi^{(\pm)}(\vec{x}) = \psi(\vec{x}) \exp\left(\frac{ie}{\hbar c} f^{(\pm)}\right)$$

with the gauge function $f^{(\pm)} = \pm 2g \arctan x_2/x_1$.

The variables in eq. (1) are separated in spherical and parabolic coordinates.

In the spherical coordinates

$$x_1 = r\sin\theta\cos\varphi, \qquad x_2 = r\sin\theta\sin\varphi, \qquad x_3 = r\cos\theta \tag{2}$$

the wave function of the dyon–dyon system is of the form [7]

$$\psi^{(s)}_{nkm}(r,\theta,\varphi) = R^{(s)}_{nkm}(r) Z^{(s)}_{km}(\theta) \frac{e^{im\varphi}}{\sqrt{2\pi}}$$

where the functions $Z^{(s)}_{km}(\theta)$ and $R^{(s)}_{nkm}(r)$ normalized by the condition

$$\int_0^\pi \sin\theta Z^{(s)}_{k'm}(\theta) Z^{(s)}_{km}(\theta) d\theta = \delta_{k'k}, \qquad \int_0^\infty r^2 \left[R^{(s)}_{nkm}(r)\right]^2 dr = 1$$

are given by the formulae

$$Z^{(s)}_{km}(\theta) = N^{(s)}_{km} (1-\cos\theta)^{\frac{|m-s|}{2}} (1+\cos\theta)^{\frac{|m+s|}{2}} P_k^{(|m-s|,|m+s|)}(\cos\theta)$$

$$R^{(s)}_{nkm}(r) = C^{(s)}_{nkm} \exp\left(-\frac{r}{r_0 n}\right) \left(\frac{2r}{r_0 n}\right)^{k+\frac{|m-s|+|m+s|}{2}}$$

$$F\left(-n+k+\frac{|m-s|+|m+s|}{2}+1; 2k+|m-s|+|m+s|+2; \frac{2r}{r_0 n}\right)$$

Here $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials; $r_0 = \hbar^2/M_0 e^2$ is the Bohr radius. The normalization constants $N^{(s)}_{km}$ and $C^{(s)}_{nkm}$ equal

$$N^{(s)}_{km} = \left[\frac{(2k+|m-s|+|m+s|+1)k!(k+|m-s|+|m+s|)!}{2^{|m-s|+|m+s|+1}\Gamma(k+|m-s|+1)\Gamma(k+|m+s|+1)}\right]^{1/2}$$

$$C^{(s)}_{nkm} = \frac{2}{n^2 r_0^{3/2}} \frac{1}{(2k+|m-s|+|m+s|+1)!} \sqrt{\frac{\left(n+k+\frac{|m-s|+|m+s|}{2}\right)!}{\left(n-k-\frac{|m-s|+|m+s|}{2}-1\right)!}}$$

Quantum numbers run over the values $n = 1, 3/2, 2, ..., k = 0, 1, ...k_{max}$, where

$$k_{max} = n - \frac{|m-s|+|m+s|}{2} - 1$$

The energy spectrum of the system is of the form

$$\epsilon^s_n = -\frac{M_0 e^4}{2\hbar^2 n^2} \tag{3}$$

In the parabolic coordinates

$$x_1 = \sqrt{\xi\eta}\cos\varphi, \quad x_2 = \sqrt{\xi\eta}\sin\varphi, \quad x_3 = \frac{1}{2}(\xi-\eta) \tag{4}$$

upon the substitution

$$\psi(\xi,\eta,\varphi) = f_1(\xi) f_2(\eta) \frac{e^{im\varphi}}{\sqrt{2\pi}}$$

the variables in (1) are separated, which results in the system of equations

$$\frac{d}{d\xi}\left(\xi\frac{df_1}{d\xi}\right)+\left[\frac{M_0\epsilon^s}{2\hbar^2}\xi-\frac{(m+s)^2}{4\xi}+\beta_1\right]f_1=0$$

$$\frac{d}{d\eta}\left(\eta\frac{df_2}{d\eta}\right)+\left[\frac{M_0\epsilon^s}{2\hbar^2}\eta-\frac{(m-s)^2}{4\eta}+\beta_2\right]f_2=0$$

where

$$\beta_1+\beta_2=\frac{M_0e^2}{\hbar^2} \tag{5}$$

At $s=0$, these equations coincide with the equations for a hydrogen atom in the parabolic coordinates [8], and consequently,

$$\psi^{(s)}_{n_1n_2m}(\xi,\eta,\varphi)=\frac{\sqrt{2}}{n^2r_0^{3/2}}f_{n_1,m+s}(\xi)f_{n_2,m-s}(\eta)\frac{e^{im\varphi}}{\sqrt{2\pi}}$$

where

$$f_{pq}(x)=\frac{1}{\Gamma(|q|+1)}\sqrt{\frac{\Gamma(p+|q|+1)}{p!}}\exp\left(-\frac{x}{2r_0n}\right)\left(\frac{x}{r_0n}\right)^{\frac{|q|}{2}}F\left(-p;|q|+1;\frac{x}{r_0n}\right)$$

Here n_1 and n_2 are non-negative integers

$$n_1=-\frac{|m+s|+1}{2}+\frac{\hbar}{\sqrt{-2M_0\epsilon^s}}\beta_1,\qquad n_2=-\frac{|m-s|+1}{2}+\frac{\hbar}{\sqrt{-2M_0\epsilon^s}}\beta_2$$

from which and (3), (5) it follows that the parabolic quantum numbers n_1, n_2, m and s are connected with the principal quantum number n as follows:

$$n=n_1+n_2+\frac{|m-s|+|m+s|}{2}+1 \tag{6}$$

3 Park–Tarter Generalized Matrix

We write the searched expansion in the form

$$\psi^{(s)}_{n_1n_2m}(\xi,\eta,\varphi)=\sum_{k=0}^{k_{max}}T^{(s)}_{n_1n_2km}\psi^{(s)}_{nkm}(r,\theta,\varphi) \tag{7}$$

Our purpose is to calculate the coefficients $T^{(s)}_{n_1n_2km}$, i.e. the Park–Tarter generalized matrix. The usual Park–Tarter matrix is the matrix $T^{(s)}_{n_1n_2km}$ at $s=0$.

We substitute

$$\xi=r(1+\cos\theta),\qquad \eta=r(1-\cos\theta),$$

into the left-hand side of expansion (7), let r tend to infinity, take the formula

$$F(-n;c;x)\sim(-1)^n\frac{\Gamma(c)}{\Gamma(c+n)}x^n,\qquad (x\to\infty)$$

and the orthogonality condition for the function $Z_{km}^{(s)}$ into account. All this leads to the formula

$$T_{n_1n_2km}^{(s)} = (-1)^k B_{n_1n_2km}^{(s)} I_{n_1n_2km}^{(s)}$$

where

$$B_{n_1n_2km}^{(s)} = \sqrt{\frac{(2k+|m-s|+|m+s|+1)k!(k+|m-s|+|m+s|)!}{2^{2n+|m-s|+|m+s|}\Gamma(k+|m-s|+1)\Gamma(k+|m+s|+1)}}$$

$$\left[\frac{\left(n-k-\frac{|m-s|+|m+s|}{2}-1\right)!\left(n+k+\frac{|m-s|+|m+s|}{2}\right)!}{(n_1)!(n_2)!\Gamma(n_1+|m+s|+1)\Gamma(n_2+|m-s|+1)}\right]^{1/2}$$

and the second factor is equal to the integral

$$I_{n_1n_2km}^{(s)} = \int_{-1}^{1}(1-x)^{n_2+|m-s|}(1+x)^{n_1+|m+s|}P_k^{(|m-s|,|m+s|)}(x)dx$$

Then taking advantage of the Rodrigues formula [9]

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]$$

and the integral representation for the Clebsch–Gordan coefficients [10]

$$C_{a\alpha;b\beta}^{c\gamma} = \delta_{\alpha+\beta=\gamma}\left[\frac{(2c+1)(J+1)!(J-2c)!(c+\gamma)!}{(J-2a)!(J-2b)!(a-\alpha)!(a+\alpha)!(b-\beta)!(b+\beta)!(c-\gamma)!}\right]^{1/2}$$

$$\frac{(-1)^{a-c+\beta}}{2^{J+1}}\int_{-1}^{1}(1-x)^{a-\alpha}(1+x)^{b-\beta}\frac{d^{c-\gamma}}{dx^{c-\gamma}}\left[(1-x)^{J-2a}(1+x)^{J-2b}\right]dx$$

$(J = a+b+c)$, we obtain

$$T_{n_1n_2ms}^{mk} = (-1)^{n_2+k}C_{a\alpha;b\beta}^{c\gamma} \tag{8}$$

where

$$a = \frac{n_1+n_2+|m+s|}{2}, \quad b = \frac{n_1+n_2+|m-s|}{2}, \quad c = k+\frac{|m-s|+|m+s|}{2}$$

$$\alpha = \frac{n_1-n_2+|m+s|}{2}, \quad \beta = \frac{n_2-n_1+|m-s|}{2}, \quad \gamma = \frac{|m-s|+|m+s|}{2}$$

At $s = 0$ formula (8) turns into the Park–Tarter formula, as would be expected.

4 Dyon–Dyon System and 4D Oscillator

Let us demonstrate that if in eq. (1) we make the changes

$$s \to -i\frac{\partial}{\partial\gamma}, \qquad \psi(\vec{x}) \to \psi(\vec{x},\gamma) = \psi(\vec{x})\frac{e^{is\gamma}}{\sqrt{4\pi}} \tag{9}$$

($\gamma \in [0, 4\pi)$), it will transform into the Schroedinger equation for a 4D isotropic oscillator.

Equation (1) in the spherical coordinates is of the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right] -$$

$$\frac{2is\cos\theta}{r^2\sin^2\theta}\frac{\partial\psi}{\partial\varphi} - \frac{s^2}{r^2\sin^2\theta}\psi + \frac{2M_0}{\hbar^2}\left(\epsilon^s + \frac{e^2}{r}\right)\psi = 0 \tag{10}$$

From (9) and (10) we have

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hat{J}^2}{r^2}\right]\psi + \frac{2M_0}{\hbar^2}\left(\epsilon^s + \frac{e^2}{r}\right)\psi = 0 \tag{11}$$

where

$$\hat{J}^2 = -\left[\frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\left(\sin\beta\frac{\partial}{\partial\beta}\right) + \frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} - 2\cos\beta\frac{\partial^2}{\partial\alpha\partial\gamma} + \frac{\partial^2}{\partial\gamma^2}\right)\right]$$

Here we change the notation: $\beta = \theta$ and $\alpha = \varphi$. If we now pass from the coordinates r, α, β, γ to the coordinates

$$u_0 + iu_1 = u\cos\frac{\beta}{2}e^{-i\frac{\alpha+\gamma}{2}}, \quad u_2 + iu_3 = u\sin\frac{\beta}{2}e^{i\frac{\alpha-\gamma}{2}} \tag{12}$$

with $u^2 = r$, take into account that

$$\frac{\partial^2}{\partial u_\mu^2} = \frac{1}{u^3}\frac{\partial}{\partial u}\left(u^3\frac{\partial}{\partial u}\right) - \frac{4}{u^2}\hat{J}^2$$

and introduce the notation

$$E = 4e^2, \qquad \epsilon^s = -\frac{M_0\omega^2}{8}$$

then equation (11) will turn into the Schrödinger equation for a 4D isotropic oscillator

$$\left[\frac{\partial^2}{\partial u_\mu^2} + \frac{2M_0}{\hbar}\left(E - \frac{M_0\omega^2u^2}{2}\right)\right]\psi(\vec{u}) = 0$$

whose energy spectrum is given by the formula

$$E_N = \hbar\omega(N+2) \tag{13}$$

Introducing the double polar coordinates

$$u_0 + iu_1 = \rho_1 e^{-i\varphi_1}, \quad u_2 + iu_3 = \rho_2 e^{i\varphi_2} \tag{14}$$

from formulae (2), (4), (12), and (14) we get the relations

$$\xi = 2\rho_1^2, \qquad \eta = 2\rho_2^2, \qquad \varphi = \varphi_1 + \varphi_2, \qquad \gamma = \varphi_1 - \varphi_2$$

which lead to the formulae

$$\psi_{NJM_1M_2}(u, \alpha, \beta, \gamma) = 4n\sqrt{\frac{2}{\lambda}}\delta_{n,\frac{N}{2}+1}\delta_{k,J-\frac{|M_1-M_2|+|M_1+M_2|}{2}}\delta_{m,M_1}\delta_{s,M_2}\psi_{nkms}(r, \theta, \varphi, \gamma)$$

$$\psi_{N_1N_2m_1m_2}(\rho_1, \rho_2, \varphi_1, \varphi_2) = 4n\sqrt{\frac{2}{\lambda}}\delta_{n_1,N_1}\delta_{n_2,N_2}\delta_{m,\frac{m_1+m_2}{2}}\delta_{s,\frac{m_1-m_2}{2}}\psi_{n_1n_2ms}(\xi, \eta, \varphi, \gamma)$$

generalizing the earlier results [6, 11].

Now we are able to write the expansion [6]

$$\psi_{N_1N_2m_1m_2}(\rho_1, \rho_2, \varphi_1, \varphi_2) = \sum_{J=J_{min}}^{N/2} W_{N_1N_2m_1m_2}^{NJM_1M_2}\psi_{NJM_1M_2}(u, \alpha, \beta, \gamma) \tag{15}$$

where

$$W_{N_1N_2m_1m_2}^{NJM_1M_2} = e^{i\pi\Phi}C_{a_0,\alpha_0;b_0,\beta_0}^{c_0,\gamma_0} \tag{16}$$

$$a_0 = \frac{N + |m_1| - |m_2|}{4}, \qquad b_0 = \frac{N - |m_1| + |m_2|}{4}, \qquad c_0 = J$$

$$\alpha_0 = \frac{N + |m_1| - |m_2|}{4} - N_2, \quad \beta_0 = \frac{N - |m_1| + |m_2|}{4} - N_1, \quad \gamma_0 = \frac{|m_1| + |m_2|}{2}$$

The lower limit of summation in (15) and quantity Φ are given by the expressions

$$J_{min} = \frac{1}{2}\left(|M_1 - M_2| + |M_1 + M_2|\right)$$

$$\Phi = N_2 + J - \frac{|m_1| + |m_2|}{2} - \frac{m_2 + |m_2|}{2}$$

We conclude with the following two comments:

(a) Using formulae (2) and (12) and considering that $r = u^2, \theta = \beta, \varphi = \alpha$, one can easily show that

$$\begin{aligned} x_1 &= 2(u_0u_2 + u_1u_3) \\ x_2 &= 2(u_0u_3 - u_1u_2) \\ x_3 &= u_0^2 + u_1^2 - u_2^2 - u_3^2 \\ \gamma &= \frac{i}{2}\ln\frac{(u_0 + iu_1)(u_2 + iu_3)}{(u_0 - iu_1)(u_2 - iu_3)} \end{aligned}$$

The first three lines are the transformation $\mathbb{R}^4 \to \mathbb{R}^3$ suggested by Kustaanheimo and Stiefel for the regularization of equations of celestial mechanics [5]. Later, this transformation found other applications, as well [12, 13]. This transformation supplemented with the coordinate γ was used for the "synthesis" of the dyon–dyon system from the 4D isotropic oscillator [4].

(b) It is known [6] that diagonal ($m_1 = m_2$) elements of the matrix $W^{NJM_1M_2}_{N_1N_2m_1m_2}$ with N even coincide with the Park–Tarter matrix. From formula (16) it follows that the remaining elements of the matrix $W^{NJM_1M_2}_{N_1N_2m_1m_2}$ have also a physical meaning: these are elements of the generalized Park–Tarter matrix for the dyon–dyon system.

5 Degeneracy of the Energy Levels

Let us discuss the problem of multiplicity of degeneration of the energy levels (3) and (13). From formula (6) it follows that at fixed n, m and s the energy levels are degenerate with the multiplicity

$$g^s_{nm} = n - \frac{|m-s| + |m+s|}{2}$$

For $s \geq 0$ the multiplicity of degeneration of levels (3) at fixed s and n is

$$g^s_n = \sum_{|m|\geq[s]} g^s_{nm} + \sum_{|m|\leq[s]-1} g^s_{nm}$$

where the upper limit of summation is determined from the condition $g^s_{nm} \geq 0$,

$$|m-s| + |m+s| \leq 2n-2$$

Therefore,

$$g^s_n = \sum_{m=-[s]+1}^{[s]-1} (n-s) + 2\sum_{m=[s]}^{[n]-1} (n-m) = (n-s)(n+s) \tag{17}$$

The same result follows from analogous computations also when $s < 0$.

The quantum numbers s and n in formula (17) assume simultaneously either integer or half-integer values, and thus, we have

$$g_n = \sum_{s=-n+1}^{n-1} g^s_n = \frac{1}{3}n(2n-1)(2n+1)$$

where g_n stands for the multiplicity of degeneration of the energy levels (13) of the 4d oscillator. Since $N = 2n-2$, we arrive at the known result

$$g_N = \frac{1}{6}(N+1)(N+2)(N+3)$$

Acknowledgments

We are sincerely grateful to G.S.Pogosyan for useful discussions.

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Received by Publishing Department
on October 11, 1996.

Мардоян Л.Г., Сисакян А.Н., Тер-Антонян В.М. E2-96-373
Матрица Парка—Тартера для системы дион-дион

Обсуждается проблема разделения переменных в системе дион-дион. Найдено линейное преобразование, связывающее фундаментальные базисы этой системы. Проведено сравнение системы дион-дион с 4D изотропным осциллятором.

Работа выполнена в Лаборатории теоретической физики им.Н.Н.Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1996

Mardoyan L.G., Sissakian A.N., Ter-Antonyan V.M. E2-96-373
Park—Tarter Matrix for a Dyon-Dyon System

The problem of separation of variables in a dyon-dyon system is discussed. A linear transformation is obtained between fundamental bases of this system. Comparison of the dyon-dyon system with a 4D isotropic oscillator is carried out.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1996

Макет Т.Е.Попеко

Подписано в печать 14.12.96
Формат 60 × 90/16. Офсетная печать. Уч.-изд. листов 1,14
Тираж 460. Заказ 49568. Цена 1368 р.

Издательский отдел Объединенного института ядерных исследований
Дубна Московской области