Classical/quantum integrability in AdS/CFT

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Motivation and plan

• *Main motivation:* quest for the integrable structure of the superconformal $\mathcal{N} = 4$ SYM$_4$ theory at large N.

• *Hope:* To relate the recently found 1- and 2-loop (and may be all-loop) quantum integrability of the SYM dilaton operator with the all-loop classical integrability of the string on the $AdS_5 \times S^5$ background.

Plan:

• Basics about the SYM operators and the dilaton "Hamiltonian" (w.r.t. RG "time") in the $SU(2)$ subsector.

• 2-loop Bethe ansatz for the SYM dilaton Hamiltonian in the subsector of 2 chiral scalars and complete solution in the "classical" limit of long operators.

• Basics about the AdS/CFT correspondence of SYM operators and strings on $AdS_5 \times S^5$.

• Complete multi-zone solution of the classical string sigma-model on $S^3 \times R_t$ (classical=BMN limit).

• Full 2-loop correspondence of SYM and string results.

Large N superconformal $\mathcal{N}=4$ SYM$_4$

$$S = \frac{\text{Tr}}{4}\int d^4x\Big(-F_{\mu\nu}^2 - (\nabla_\mu\Phi_a)^2 + 2i\bar{\chi}^a[\not\nabla,\chi_a]$$

$$-[\Phi_a,\Phi_b]^2 + 2i\bar{\chi}[\Gamma^a\Phi_a,\chi]\Big),$$

$\nabla_\mu = \partial_\mu + g_{YM}[A_\mu,\ldots]$; fields in Adj($su(N)$): A_μ, Majorana-Weyl spinors χ, $O(6)\sim SU(4)$ scalars:

$$X = \Phi_1 + i\Phi_2, \qquad Y = \Phi_4 + i\Phi_3, \qquad Z = \Phi_5 + i\Phi_6.$$

• Local operators belong to representations of superconformal algebra $su(2,2|4)$:

$$\mathcal{O}(x) = \text{Tr}\,\left(\nabla^m\ F^n\ \bar{\chi}^k\chi^k\ X^{J_1}Y^{J_2}Z^{J_3}\bar{X}^{J_1'}\bar{Y}^{J_2'}\bar{Z}^{J_3'}\right) + \text{perm.}$$

• Dilaton operator $\hat{D} = \hat{D}^{(0)} + \lambda\hat{D}^{(2)} + \lambda^2\hat{D}^{(4)} + \ldots$ ($\lambda = Ng_{YM}^2$ - 't Hooft coupling):

$$\mathcal{O}(x/\Lambda) = \Lambda^{\hat{D}}\mathcal{O}(x) = \Lambda^{\hat{D}^{(0)}}\left(1 + \lambda\log\Lambda\hat{D}^{(2)} + \ldots\right)\mathcal{O}(x)$$

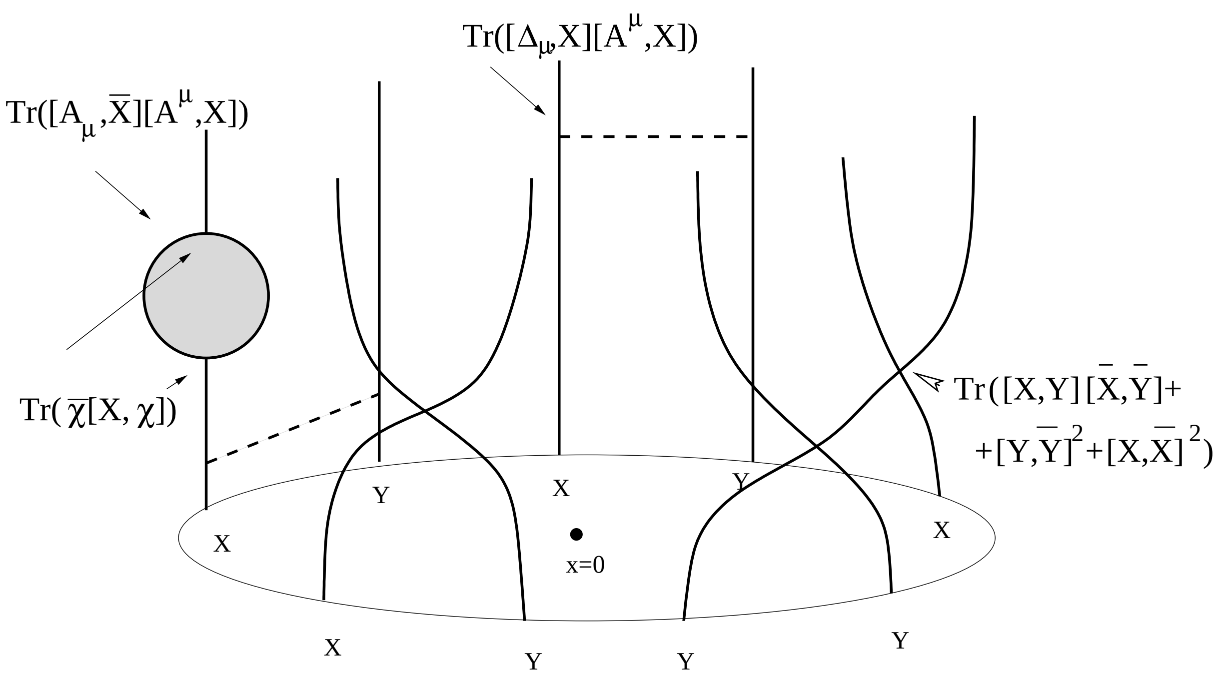
• Conf. dimensions $\Delta = \Delta^{(0)} + \lambda\Delta^{(2)} + \lambda^2\Delta^{(4)} + \ldots$ are eigenvalues of "hamiltonian" $\hat{D}$ ("time" $\sim \log\Lambda$).

Dilaton as a spin chain hamiltonian

- $SU(2)$-sector ($X = \uparrow, \quad Y = \downarrow$), chain of length L:

$$\mathcal{O} = \mathrm{Tr}\ (XXXYYXYY\ldots) = |\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\ldots\rangle \in \left(\mathbf{C}^2\right)^{\otimes L}.$$

- Calculation of $\hat{D}$ by perturbation theory: point splitting and renormalization:



$$\hat{D} = L + \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(1 - \hat{P}_{l,l+1}\right) + O(\lambda^2)$$

$\hat{P}_{l,l+1} = \frac{1}{2}\left(1 + \sigma_l \cdot \sigma_{l+1}\right)$ - permutation operator.

Dilaton at two and more loops

• $\hat{D}$ is known in 1-loop (the standard XXX-chain) [Minahan,Zarembo '02] and 2 loops [Beisert,Kristjansen,Staudacher '02], conjectured up to 5 loops (from integrability and BMN scaling) [Beisert,Dippel,Staudacher '04]

$$\hat{D} = L + \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} (1 - \sigma_l \cdot \sigma_{l+1}) +$$

$$\left(\frac{\lambda}{16\pi^2}\right)^2 \sum_{l=1}^{L} ((1 - \sigma_l \cdot \sigma_{l+2}) - 4\,(1 - \sigma_l \cdot \sigma_{l+1})) + O(\lambda^3)$$

• k-loop $\hat{D}$ has interactions up to k neighbors.

• $\left(\hat{D} - L\right) \mathrm{tr}\,(X^L) = 0$, as a consequence of SUSY.

• **Integrability**: *proven at two loops and is a great hope in all loops!*

• Proposal [Serban,Staudacher 03']: the $SU(2)$ sector is described in *all loops* by the integrable nonlocal *Inozemtsev chain*

$\hat{H} = \sum_{k>l=1}^{L} h_{|k-l|}\,(L, \kappa(\lambda))\ \ \sigma_l \cdot \sigma_k$

where $h_k\,(L, \kappa)$ is related to Weierstrass function.

Describes known 2 loops and (if integrable) 3 loops.

Solvable by **Bethe ansatz**!

Anomalous dimensions at 2 loops

- To get the complete set of 2-loop anomalous dimensions $\gamma = \Delta - L$ we find the combinations of one-trace operators satisfying the "Schroedinger eq." $\hat{D}\mathcal{O} = \Delta\mathcal{O}$, where

$$\mathcal{O} = \mathrm{Tr}\ \left(X^J Y^{L-J} + \text{permut. with coeff.}\right)$$

- "2-loops": BA energy = anomalous dimension

$$\gamma = \sum_{j=1}^{J}\left[\frac{\lambda}{8\pi^2}\frac{1}{u_j^2+1/4} + \frac{3\lambda^2}{128\pi^4}\frac{1}{(u_j^2+1/4)^2} - \frac{\lambda^2}{128\pi^4}\frac{1}{(u_j^2+1/4)^3}\right] + \cdots$$

- The rapidities u_j satisfy BA eqs.

$$e^{ipL} = \left(\frac{u_j + i/2 - \frac{\lambda}{8\pi^2}\frac{u_j}{u_j^2+1/4}}{u_j - i/2 - \frac{\lambda}{8\pi^2}\frac{u_j}{u_j^2+1/4}}\right)^L = \prod_{k=1(k\neq j)}^{J}\frac{u_j - u_k + i}{u_j - u_k - i}$$

- Cyclicity of trace ("zero total momentum")

$$\prod_{j=1}^{J}\left(\frac{u_j + i/2 - \frac{\lambda}{8\pi^2}\frac{u_j}{u_j^2+1/4}}{u_j - i/2 - \frac{\lambda}{8\pi^2}\frac{u_j}{u_j^2+1/4}}\right)^L = 1$$

"Classical" limit of long operators

• In this ("Gaudin") limit $L, J \sim \infty$, the rapidities scale as $u \sim Lx$, and we introduce their resolvent and density on a system of supports $x \in C_k,\ k = 1, \cdots, K$ in the complex plane x: $G_g(x) = \int \frac{dy \rho_g(y)}{x-y}$, with the normalization

$$G_g(x) \sim \frac{J/L}{x} + O(1/x^2), \quad x \to \infty.$$

• Up to 3 loops in $T = \frac{\lambda}{16\pi^2 L^2}$ the BA eqs. reduce to the following Riemann-Hilbert eqs. at supports $x \in \mathbf{C}_l$

$$G_g(x+i0)+G_g(x-i0) = \frac{1}{x}+\frac{2T}{x^3}+\frac{6T^2}{x^5}+2\pi n_l, \quad n_l = 0 \pm 1, \pm 2 \dots$$

• Cyclicity reduces to the condition at $x = 0$:

$$2\pi m = -G_g(0) - T G_g''(0) - \frac{T^2}{4} G_g^{(4)}(0), \quad n_l = 0 \pm 1, \pm 2 \dots$$

• Dimensions from the behaviour at $x = 0$:

$$\frac{\Delta - L}{2TL} = -G_g'(0) - \frac{T}{2} G_g'''(0) - \frac{1}{6} T^2 G_g^{(5)}(0)$$

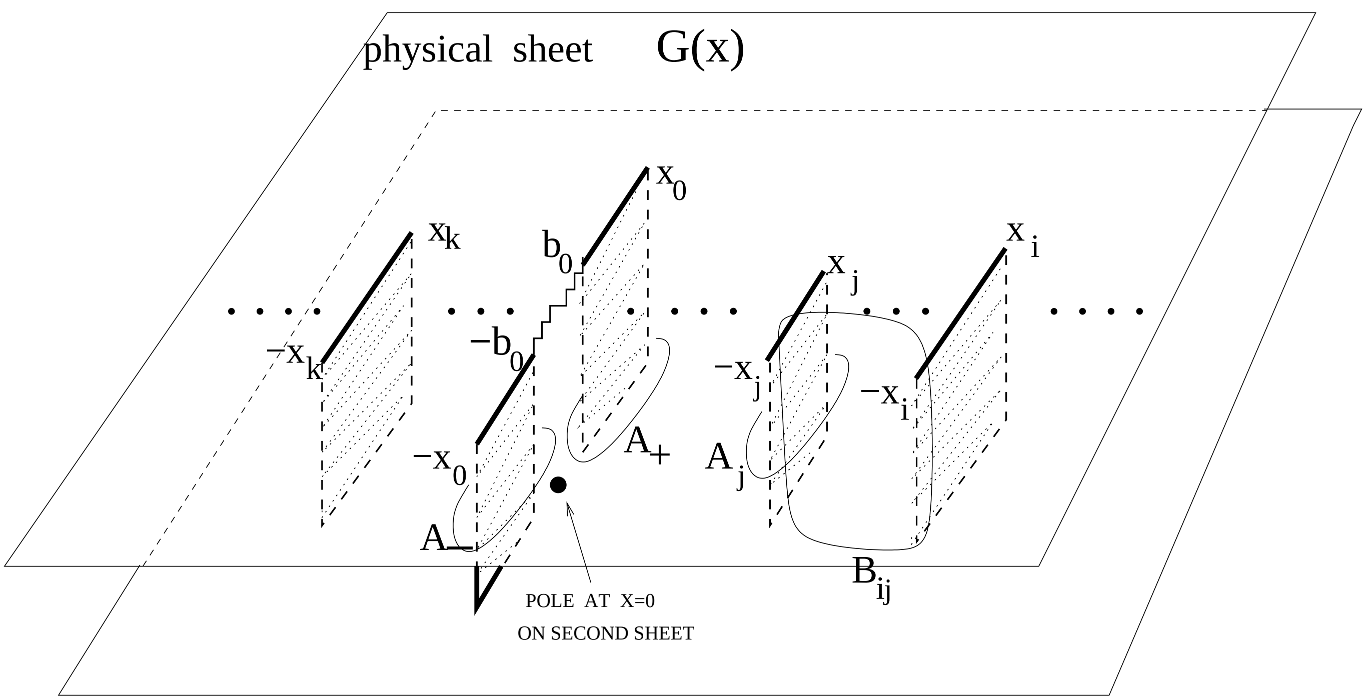
General solution for long operators

• The *quasi-momentum* $p(x) = G(x) - \frac{1}{x} + \frac{2T}{x^3} + \frac{6T^2}{x^5}$ has the known behavior at $x \to \infty$, the known poles at $x = 0$ and satisfies the BA eq.

$$p(x+i0) + p(x-i0) = 2\pi n_l, \quad n_l, \quad x \in C_l$$

Hence $p(x)$ is a double-valued function on the hyperelliptic Riemann surface Σ

$$\Sigma : \quad y^2 = R_{2K}(x) = x^{2K} + r_1 x^{2K-1} + \ldots + r_{2K} = \prod_{j=1}^{2K} (x - \mathrm{x}_j)$$



- The 3-loop solution can be written as

$$dp = \frac{dx}{y(x)} \sum_{k=-5}^{K-1} a_k x^{k-1}$$

- Single-valuedness of $p(x)$:

$$\oint_{A_l} dp = 0 \qquad l = 1, \ldots, K-1$$

- BA eqs. become integer B-period conditions:

$$\oint_{B_{jk}} dp = 2\pi(n_j - n_k) \qquad n_j, n_k = 1, \ldots, K-1$$

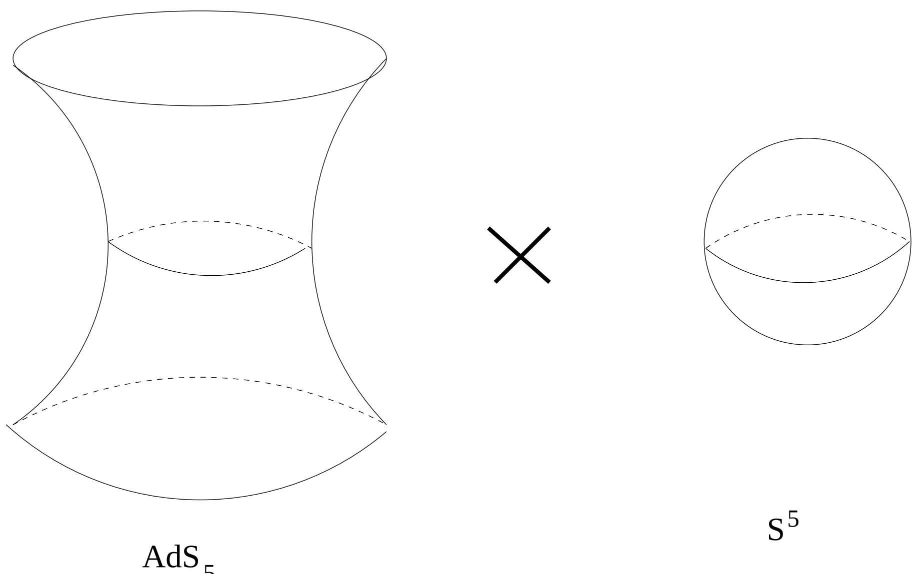
- Given the filling fractions $S_i = \int_{C_l} dx \rho_g(x)$ the quasi-momentum is defined unambiguously, and the zero momentum condition reads

$$\sum_{l=1}^{K} n_j S_j = m$$

Correspondence to strings at AdS$_5 \times S^5$

[Maldacena '98], [Gubser,Klebanov, Polyakov '98], [Berenstein,Maldacena Nastase '02], [Metsaev,Tseytlin '02], [Frolov,Tseytlin '02],...

• $\mathcal{N} = 4$ SYM theory is dual to the Green-Schwarz string theory on AdS$_5 \times S^5$ of the radius $R = \frac{\sqrt{\lambda}}{\alpha'}$



$$\mathrm{AdS}_5: \quad -X_{-1}^2 - X_0^2 + X_7^2 + X_8^2 + X_9^2 + X_{10}^2 = -R^2$$

$$S^5: \quad X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = R^2$$

• The radial coordinate z and the Lorentzian space-time x_μ of AdS are recovered from $X_{-1} + X_{10} = R/z$, $(X_0, X_7, X_8, X_9) = R\frac{x_\mu}{z}$, giving $ds^2 = R^2\,\frac{dx^2 + dz^2}{z^2}$.

SYM dual in SU_2 sector: string on $S^3 \times R_t$

$$S_\sigma = \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \int d\tau \left[(\partial_a X_i)^2 - (\partial_a X_0)^2 \right], \quad X_i X_i = 1, \ i$$

where $X_0 \in R_t$ is the global AdS time and $X_i \to S^3$-section of the full $AdS_5 \times S^5$. String is projected to a point for the rest of coordinates.

• String tension is related to YM coupling: $\frac{1}{\alpha'} = \sqrt{\lambda}$. For $\lambda \to \infty$ the string becomes classical and is exactly solvable.

• $O(3)$ σ-model is the $SU(2)$ principal chiral field:

$$S_\sigma = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[\frac{1}{2} \mathrm{Tr} j_a^2 + (\partial_a X_0)^2 \right]$$

with right current $j_a = g^{-1} \partial_a g = \frac{1}{2i} j_a^A \sigma^A$, left current $l_a = g \partial_a g^{-1}$, and

$$g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix} \equiv \begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \in SU(2)$$

AdS/CFT dictionary

• σ-model on S^3 has a global $SU_L(2) \times SU_R(2) \in SO(4)$ symmetry, the same as the SYM scalars X, Y. Hence the conserved quantities for X, Y and Z_1, Z_2 should coincide.

• Left shifts $g \to hg$ and right shifts $g \to gh$ are generated by conserved charges

$$Q^3_{L,R} = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma \, \mathrm{Tr} \, \left(\sigma^3 g^{\pm 1} \partial_0 g^{\mp 1}\right)$$

• Under left shifts $(Z_1, -\bar{Z}_2)$ and $(Z_2, -\bar{Z}_1)$ transform as doublets, so that $Q^3_L = 1$ for Z_1 or Z_2. Hence for an operator $\mathrm{Tr}(X^{L-J} Y^J + \ldots)$

$$Q^3_L = L.$$

• Under right shifts (Z_1, Z_2) transforms as a doublet, so that Z_1 has $Q^3_R = 1$ and Z_2 has $Q^3_R = -1$. Consequently,

$$Q^3_R = L - 2J.$$

• Virasoro conditions in the gauge $X_0 = \kappa\tau$:

$$(\partial_\pm X_i)^2 = (\partial_\pm X_0)^2 = \kappa^2, \quad \sigma_\pm = \frac{1}{2}(\tau \pm \sigma).$$

• Dimension of an operator is dual to the energy of the string solution generated by the global time translations [Gubser,Polyakov,Klebanov '98]:

$$\Delta = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\, \partial_\tau X_0 = \sqrt{\lambda}\,\kappa,$$

Thus, the Virasoro constraints become

$$\frac{1}{2}\mathrm{Tr} j_+^2 = \frac{1}{2}\mathrm{Tr} j_-^2 = -\kappa^2\,.$$

Integrability and multi-zone solution

- Eqs. of motion

$$\partial_+ j_- + \partial_- j_+ = 0, \quad \partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0, \quad \partial_+ \partial_- X_0 = 0$$

can be rewritten as a single zero-curvature eq. through $J_\pm(x) = \frac{j_\pm}{1 \mp x}$ (x is a spectral parameter):

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0$$

- Associated linear problem:

$$\mathcal{L}\Psi = \left(\partial_\sigma + \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x}\right)\right)\Psi = 0,$$

$$\mathcal{M}\Psi = \left(\partial_\tau + \frac{1}{2}\left(\frac{j_+}{1-x} + \frac{j_-}{1+x}\right)\right)\Psi = 0.$$

- Monodromy matrix generates conserved quantities

$$\mathrm{Tr}\ \Omega(x) = \mathrm{Tr}\ \hat{P} \exp \int_0^{2\pi} d\sigma\, \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x}\right) = 2\cos p(x)$$

where $p(x)$ is the quasi-momentum (real by unitarity!).

Properties of quasi-momentum $p(x)$

- The standard asymptotic analysis yields

$$p(x) = -\frac{\pi\kappa}{x \pm 1} + \dots \qquad (x \to \mp 1).$$

- At $x \to \infty$ $\Lambda = \partial_\sigma + j_0/x + \dots$, and

$$\text{Tr}\ \Omega \simeq 2 + \frac{1}{2x^2}\int_0^{2\pi} d\sigma_1 d\sigma_2\, \text{Tr}\ j_0(\sigma_1) j_0(\sigma_2) = 2 - \frac{4\pi^2 Q_R^2}{\lambda x^2}$$

$$\text{or} \qquad p(x) = -\frac{2\pi(L-2J)}{\sqrt{\lambda}\, x} + \dots \qquad (x \to \infty)$$

- At $x \to 0$, $\mathcal{L} = \partial_\sigma + j_1 - x j_0 + \dots$, which can be written as $\mathcal{L} = g^{-1}(\partial_\sigma - x l_0 + \dots)g$. Then,

$$\Omega(x) \simeq g^{-1}(2\pi)\hat{P}\exp\left(-x\int_0^{2\pi} d\sigma\, l_0 + \dots\right) g(0).$$

Because of the periodicity of $g(\sigma)$, $\Omega(0) = 1$ and $p(0) = 2\pi m$. Expanding further we find

$$p(x) \sim 2\pi m + \frac{2\pi L}{\sqrt{\lambda}} x + O(x^2) \qquad (x \to 0).$$

• Two lineraly independent solutions of $\mathcal{L}\Psi_\pm = 0$ can be chosen quasi-periodic:
$\Psi_\pm(x, \sigma + 2\pi) = e^{\pm ip(x)}\Psi_\pm(x, \sigma)$,
since $\Psi(x, \sigma + 2\pi) = \Psi(x, \sigma)\Omega(x)$.
$\Psi_\pm(x, 0)$ can be viewed as two branches of the same analytical function on the double cover (Riemann surface) of the complex plain x.

• Quasi-momentum can be complex on a set of disjoint linear supports C_k, or cuts on a hyperelliptic surface. At the branch-points defined by the quadratic eq.

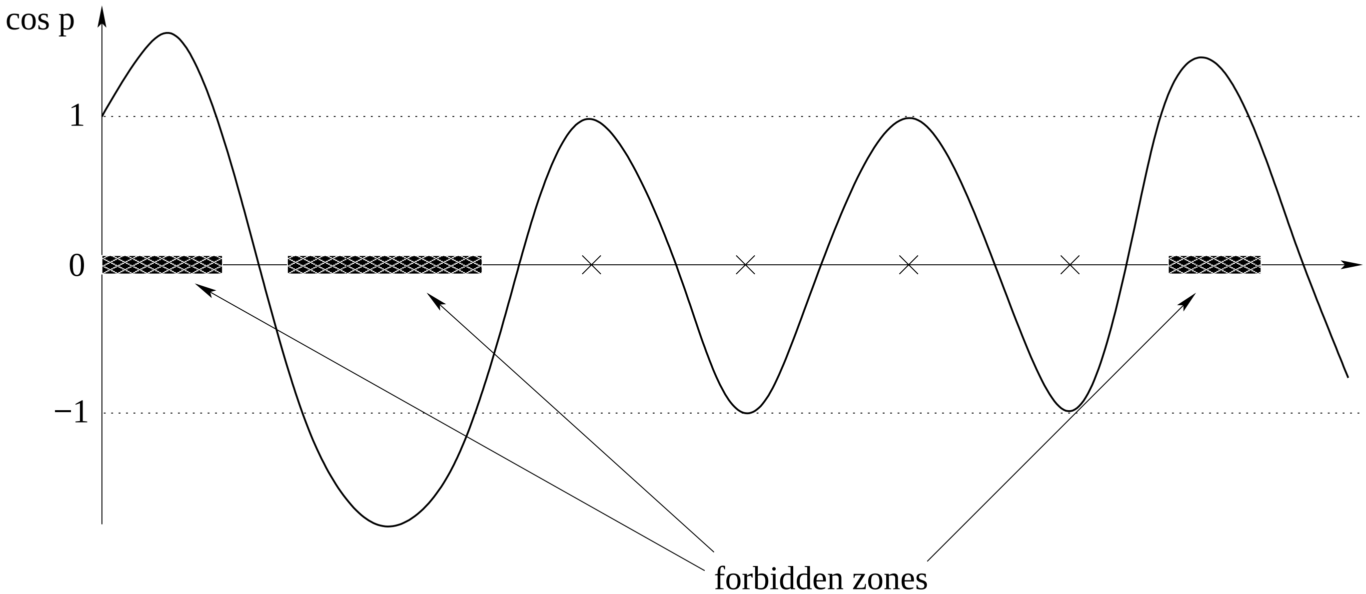
$$e^{ip} + e^{-ip} = \mathrm{Tr}\ \Omega(x)$$

where $\Omega(x)$ is an entire function, $\Psi_\pm(x, 0)$ coincident. Two branches $e^{\pm ip(x)} = e^{ip(x \pm i0)}$ correspond to two eigenvalues of the monodromy matrix $\Omega(x)$.

• From unitarity $\mathrm{Det}\Omega(\mathrm{x}) = \mathrm{e}^{\mathrm{ip(x-i0)}}\mathrm{e}^{\mathrm{ip(x+i0)}} = 1$, or

$$p(x + i0) + p(x - i0) = 2\pi n_k, \qquad x \in C_k$$

$Tr\Omega(x)$ **at real** $\cos p(x)$



Associated Riemann-Hilbert problem

• The only singularities of $p(x)$ are cuts and poles at $x = \pm 1$. Subtracting them and introducing the "resolvent" and the "density" $G_s(x) = \int \frac{dy \rho_s(x)}{x-y}$ by the formula

$$G(x) = p(x) + \frac{\pi\kappa}{x-1} + \frac{\pi\kappa}{x+1}$$

we rewrite the unitarity eq. and the conditions at $x = 0, \infty$ as the following system:

- $$G_s(x) \sim \left(\frac{J}{L} + \frac{\Delta - L}{2TL}\right)\frac{1}{x}, \qquad x \to \infty$$
- $$G_s(x + i0) + G_s(x - i0) = \frac{\Delta}{L}\frac{x}{x^2 - T} + 2\pi n_j$$
- $$2\pi m = -G_s(0)$$
- $$\frac{\Delta - L}{2TL} = -G_s'(0)$$

Solvable problem, similar to SYM chain at $L \to \infty$.

AdS/CFT correspondence at 2 loops

• Attempting to fit σ-model and SYM results we try the following relation between the resolvents:

$$G_g(x) = G_s(x - T/x) + T\frac{G_s'(0)}{x} \simeq G_s(x) - T\frac{G_s'(x) - G_s'(0)}{x} + \dots .$$

This equation describes the map of higher charges in the sigma model to those in the gauge theory.

• We immediately find the perfect coincidence of σ-model and SYM results for the normalization

$$xG_g(x) \sim \frac{J}{L} + \frac{\Delta - L}{2TL} + G_s'(0) = \frac{J}{L}, \qquad x \to \infty$$

the zero momentum condition (periodicity)

$$2\pi m = -\left[G_s(0) - TG_s''(0)\right] + TG_s''(0) = -G_s(0)$$

and anomalous dimension

$$\frac{\Delta - L}{2TL} = -\left[G_s'(0) - \frac{6T}{2 \cdot 6}G_s''''(0)\right] + \frac{T}{2}G_s''''(0) = -G_s(0)$$

• Finally, the SYM BA eq. written in terms of $\not{G}_s(x) = \frac{1}{2}(G_s(x+i0) + G_s(x-i0))$

$$2\pi n_j + \frac{1}{x} + \frac{2T}{x^3} = 2\not{G}_g(x) = 2\not{G}_s(x) - \frac{2T}{x}\not{G}'_s(x) - \left(\frac{\Delta}{L} - 1\right)\frac{1}{x}$$

can be rewritten up to $O(T^2)$ as

$$2\pi n_j + \frac{1}{x} + \frac{2T}{x^3} = 2\not{G}_s(x - T/x)$$

Changing the variable to $z = x - T/x$ we rewrite it in the same form as for the σ-model

$$2\not{G}_s(z) = 2\pi n_j + \frac{\Delta/L}{z} + \frac{2T}{z^3} \simeq \frac{\Delta}{L}\frac{z}{z^2 - T} + 2\pi n_j$$

The two-loop equivalence of the SYM $SU(2)$ chain and the $S^3 \times R$ σ-model is thus verified for a very general class of finite gap solutions of σ-model, matching the multi-cut solutions for SYM.

Before it was done only for the restricted class of symmetric two-cut solutions in [Serban,Staudacher '04],[Arutyunov,Staudacher '04].

Conlcusion and prospects

The problems start occuring at ≥ 3 loops:

• The limit of small BMN parameter $T = \frac{\lambda}{16\pi L^2}$ of the σ-model does not reproduce the SYM results if we assume an integrable 3-loop dilaton operator.

• The quantum $1/L$ string correctons to $O(T^3)$ terms do not agree with SYM results [Callan et al. '03-'04].

Possible reasons:

- A difference of the exact result for the hopefully integrable SYM in $1 << \lambda << L^2$ and $\lambda << 1 << L^2$ regimes?

- Wrong σ-model Lagrangian?

- No integrability at 3 loops? Difficult to check... (though an arguments in favor of it exists [Beisert '03].

Some problems

• One-loop solution for dimensions of the full SYM theory, for any long operators of integrable $su(2,2|4))$ chain of [Beisert '03,Beisert,Staudacher '03]

• Find an integrable quantum spin model having σ-model as its classical limit.

• Integrability in the usual YM?
[Lipatov '94], [Faddeev,Korchemski '95],
[Ferretti,Heise,Zarembo '04].

Last month event: Inspired by our 2-loop change of variables from SYM to σ-model, [Beisert,Dippel,Staudacher '04] proposed the *all loop* BA eqs. for the SYM spin chain. Resembles the σ-model eqs. but differs from them in details:

$$2\not{G}_\sigma(x) = \frac{x}{x^2 - T}\left(1 - T G'_\sigma(0)\right) + 2\pi n_k$$

$$2\not{G}_g(x) = \frac{x}{x^2 - T} - \left(G_g(T/x) - G_g(0)\right) + 2\pi n_k$$