DEFORMATIONS OF (SUPER)SYMMETRY IN NONCOMMUTATIVE FIELD THEORY

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INTRODUCTION

The simplest type of the space-time noncommutativity is connected with the following relation for the coordinate operators $\hat{x}^m$

$$C^{mn}_\star \equiv \hat{x}^m \star \hat{x}^n - \hat{x}^n \star \hat{x}^m - i\vartheta^{mn} = 0$$

where ϑ^{mn} $(m, n = 1, 2, 3, 4)$ is some constant tensor specifying the deformation of the commutative 4D coordinates. Noncommutative algebra of fields $\hat{f}(\hat{x})$ on this deformed space-time is formally analogous to the Weyl algebra on the quantized phase space $\hat{x}, \hat{p}, \quad [\hat{p}, \hat{x}] = i\hbar$. The Weyl ordering of the operator field $\hat{f}(\hat{x})$ uses the decomposition in terms of completely symmetrized monomials $\hat{x}^{(m_1} \star \ldots \star \hat{x}^{m_n)}$

$$\hat{f}(\hat{x}) = \sum_{n=0}^{\infty} c_{m_1 \ldots m_n} \hat{x}^{(m_1} \star \ldots \star \hat{x}^{m_n)}$$

The popular realization of this noncommutative algebra $A_\star$ can be defined on ordinary smooth functions $f(x), g(x)$ on R^4 using the following pseudolocal representation of the $\star$-product :

$$f \star g = f e^P g = fg + \tfrac{i}{2}\vartheta^{mn}\partial_m f \partial_n g - \tfrac{1}{8}\vartheta^{mn}\vartheta^{rs}\partial_m\partial_r f \partial_n \partial_s g + \ldots,$$
$$= fg + \partial_m[\tfrac{i}{2}\vartheta^{mn} f \partial_n g - \tfrac{1}{8}\vartheta_{mn}\vartheta^{rs}\partial_r f \partial_n \partial_s g + \ldots]$$

where $\partial_m = \partial/\partial x_m$ are the partial drivatives on R^4. All products of the functions and their derivatives in the right-hand side are commutative. In this representation, the basic commutator relation is satisfied identically

$$x^m \star x^n - x^n \star x^m - i\vartheta^{mn} \equiv 0$$

It is convenient to analyze the noncommutative field theory just in this pseudolocal representation, although all properties of the theory can be reformulated in the operator representation.

Let us consider the simple ϑ-noncommutative interaction of the real scalar field $\phi(x)$ using the $\star$-product algebra $A_\star$

$$S_\star = \int d^4x L_\star(\phi) = \tfrac{1}{2}\int d^4x(\eta^{mn}\partial_m\phi \star \partial_n\phi - M^2\phi^2_\star - \lambda\phi^4_\star)$$

where M and λ are the mass and coupling constant, respectively, and η^{mn} is the metric of the Minkowski space. In the pseudolocal representation , the quadratic free terms of this action are not deformed. The nonlinear interaction depends manifestly on ϑ_{mn}, so it is not invariant with respect to the standard local Lorentz transformation

$$\delta_\omega\phi = \phi'(x) - \phi(x) = -\tfrac{1}{2}\omega^{mn}M_{mn}\phi = -\omega^m{}_n x^n \partial_m \phi,$$

$$M_{mn} = x_n\partial_m - x_m\partial_n, \quad [\partial_r, M_{mn}] = \eta_{rn}\partial_m - \eta_{rm}\partial_n$$
$$M_\omega = \tfrac{1}{2}\omega^{mn}M_{mn} = \omega^{mn}x_n\partial_m, \quad \omega^m{}_n = \eta_{nr}\omega^{mr}$$

where ω^{mn} are the corresponding infinitesimal parameters.

All models of the ϑ-noncommutative field theory use the following basic rule:
A) Primary fields are local representations of the Lorentz group and interactions of these primary fields contain the same $\star$-product.

The transformation of $\partial_m\phi$ is

$$\delta_\omega\partial_m\phi = \partial_m\delta_\omega\phi = -\tfrac{1}{2}\omega^{rs}M_{rs}\partial_m\phi - \tfrac{1}{2}\omega^{rs}[\partial_m, M_{rs}]\phi$$
$$= -\tfrac{1}{2}\omega^{rs}M_{rs}\partial_m\phi + \eta_{ms}\omega^{sr}\partial_r\phi = -M_\omega\partial_m\phi - \omega^r{}_m\partial_r\phi$$
$$\delta_\omega x^m = \omega^m{}_n x^n$$

Noncovariance of the noncommutative product with respect to the Lorentz group.

$$\delta_\omega(f\star g) = -(M_\omega f\star g) - (f\star M_\omega g) \neq -M_\omega(f\star g)$$

The finite local transformation of the scalar field has the following form:

$$\phi'(x) = \phi(\tilde{x}) = e^{-M_\omega}e^{-P(c)}\phi(x), \quad \tilde{x}^m = e^{-M(\omega)}(x^m - c^m)$$

where $P(c) = c^m\partial_m$. These formulas define universal enveloping bialgebra Poincaré $U(P_m, M_{mn})$. Products or commutators of generators can be calculated in the differential representation. A coproduct in the bialgebra $U(P_m, M_{mn})$ defines the map to the tensor product $U(P_m, M_{mn})\otimes U(P_m, M_{mn})$. On the generators, this coproduct has a trivial form

$$\Delta(P_m) = P_m\otimes 1 + 1\otimes P_m, \quad \Delta(M_{mn}) = M_{mn}\otimes 1 + 1\otimes M_{mn}$$

The basic properties of the coproduct

$$\Delta([M_{mn}, M_{rs}]) = [\Delta(M_{mn}), \Delta(M_{rs})],$$
$$\Delta([M_{mn}, P_r]) = [\Delta(M_{mn}), \Delta(P_r)]$$

It determine the coaction of generators on tensor products of representations

$$\Delta(M_{mn})\circ f\otimes g = M_{mn}f\otimes g + f\otimes M_{mn}g, \quad \delta_\omega(f\otimes g) = \delta_\omega f\otimes g + f\otimes\delta_\omega g.$$

A standard Leibniz rule for the infinitesimal Lorentz transformation in the commutative algebra of fields follows from the trivial coproduct $\Delta(M_{mn})$

$$\delta_\omega(fg) = \mu\circ\delta_\omega(f\otimes g) = (\delta_\omega f)g + f\delta_\omega g.$$

where the bilinear map $\mu\circ f\otimes g = fg$ defines a multiplitation $U\otimes U\to U$. This rule seems evident in the commutative field theory, however, the trivial coproduct can be deformed in the framework of quantum groups, so the rules of transformations of the noncommutative fields can be changed.

TWISTED POINCARÉ SYMMETRY

The basic point of this interpretation is connected with the twist operator acting on tensor products of functions

$$\mathcal{F} = \exp(\mathcal{P}), \quad \mathcal{P} = \tfrac{i}{2}\vartheta^{mn} P_m \otimes P_n$$

and $P_m f = \partial_m f$. The strict definition of the noncommutative product is connected with operator $\mathcal{F}$

$$f \star g \equiv \mu_\star \circ f \otimes g = \mu \circ \mathcal{F} \circ f \otimes g,$$

where μ is the multiplication map in the commutative algebra of functions $A(R^4)$, and $\mu_\star = \mu \circ \mathcal{F}$ defines the analogous map in $A_\star$. Note that the tensor product can be treated as a nonlocal product of fields in different points $f(x_1)g(x_2)$.

Let us consider generators of the Poincaré group P_m and M_{mn}. By definition, the twist-deformed Poincaré bialgebra $U_t(P_m, M_{mn})$ has the undeformed Lie algebra of generators; however, its coproduct is deformed

$$\Delta_t(P_m) = P_m \otimes 1 + 1 \otimes P_m = \Delta(P_m),$$

$$\Delta_t(M_{mn}) = \exp(-\mathcal{P})\Delta(M_{mn})\exp(\mathcal{P}) = \Delta(M_{mn})$$
$$+\tfrac{i}{2}\vartheta_{ms} P_n \otimes P^s - \tfrac{i}{2}\vartheta_{ns} P_m \otimes P^s - \tfrac{i}{2}\vartheta_{rn} P^r \otimes P_m + \tfrac{i}{2}\vartheta_{rm} P^r \otimes P_n$$

The manifest description of the local transformations of fields in the noncommutative algebra $A_\star$ was given in recent papers of the Munchen group. These authors have constructed the map between differential operators on commutative and noncommutative algebras of functions. Let $\xi = \xi^m(x)\partial_m$ is the 1-st order differential operator on $A(R^4)$, then one can build the corresponding operator $\hat{X}_\xi$ on $A_\star$

$$(\xi f) = \mu \circ (\xi^m \otimes \partial_m)(1 \otimes f) = \mu_\star \exp(-\mathcal{P})(\xi^m \otimes \partial_m)(1 \otimes f)$$
$$= \xi^m \star \partial_m f - \tfrac{i}{2}\vartheta^{rs}\partial_r \xi^m \star \partial_s \partial_m f + O(\vartheta^2) = (\hat{X}_\xi \star f)$$

The local generators of the Poincare group P_m and M_{mn} correspond to the following operators on the noncommutative algebra $A_\star$:

$$(\hat{P}_m \star f) = \partial_m f, \quad (\hat{M}_{mn} \star f) = (M_{mn} f)$$

$$\hat{M}_{mn} = x_n \star \partial_m - x_m \star \partial_n + \tfrac{i}{2}\eta_{mr}\vartheta^{rs}\partial_s\partial_n - \tfrac{i}{2}\eta_{nr}\vartheta^{rs}\partial_s\partial_m$$

Operators $\hat{P}_m$ and $\hat{M}_{mn}$ form the Lie algebra isomorphic to the standard Poincaré algebra generated by P_m and M_{mn}

$$([M_{mn}, M_{rs}]_\star) \star f = [M_{mn}, M_{rs}]f$$

$$[M_{mn}, M_{rs}] = \eta_{ms} M_{rn} + \eta_{nr} M_{sm} - \eta_{ns} M_{rm} - \eta_{mr} M_{sn}$$

Coproduct Δ_t acts on the tensor product of fields

$$-\tfrac{1}{2}\omega^{mn}\Delta(M_{mn}) \circ f \otimes g = (\delta_\omega f) \otimes g + f \otimes (\delta_\omega g)$$

$$+\tfrac{i}{2}(\omega^{mn}\vartheta_{ns}-\vartheta^{mn}\omega_{ns})\partial_m f\otimes\partial^s g$$

Applying the map $\mu_\star$ to this relation, one can prove the covariance of the noncommutative algebra $A_\star$ with respect to $U_t(P_m,M_{mn})$ transformations

$$\hat{\delta}_\omega\star(f\star g)=-\tfrac{1}{2}\omega^{mn}\mu_\star\Delta(M_{mn})\circ f\otimes g=(\delta_\omega f)\star g+f\star(\delta_\omega g)$$

$$+\tfrac{i}{2}(\omega^{mn}\vartheta_{ns}-\vartheta^{mn}\omega_{ns})\partial_m f\star\partial^s g=-\tfrac{1}{2}\omega^{mn}M_{mn}(f\star g)$$

This relation can be direcly checked using the psedolocal formula

$$M_{mn}(f\star g)=M_{mn}(f\exp(P)g)=(M_{mn}f\exp(P)g+f\exp(P)M_{mn}g$$

$$+\tfrac{i}{2}\vartheta_{ms}P_nf\exp(P)P^sg-\tfrac{i}{2}\vartheta_{ns}P_mf\exp(P)P^sg$$
$$-\tfrac{i}{2}\vartheta_{rn}P^rf\exp(P)P_mg+\tfrac{i}{2}\vartheta_{rm}P^rf\exp(P)P_ng$$

Applying the map μ to $\Delta_t(M_{mn})$ one obtains the noncovariant action of $U_t(P_m,M_{mn})$ on the commutative product of fields

$$\hat{\delta}_\omega\star(fg)=-\tfrac{1}{2}\omega^{mn}\mu\circ\Delta_t(M_{mn})\circ f\otimes g=-(M_\omega f)g-f(M_\omega g)$$

$$+\tfrac{i}{2}(\omega^{mn}\vartheta_{ns}-\vartheta^{mn}\omega_{ns})\partial_m f\partial^s g\neq-M_\omega(fg)$$

where $M_\omega=\frac{1}{2}\omega^{mn}M_{mn}=\omega^{mn}x_n\partial_m$

Transformations of different noncommutative field terms:

$$\hat{M}_\omega\star\phi^2_\star=M_\omega(\phi^2_\star)=(M_\omega\phi)\star\phi+\phi\star(M_\omega\phi)$$
$$-\tfrac{i}{2}\omega^p{}_r\vartheta^{rs}\partial_p\phi\star\partial_s\phi+\tfrac{i}{2}\omega^p{}_r\vartheta^{rs}\partial_s\phi\star\partial_p\phi$$

The 2-nd term is a full devivative and its 4D integral vanishes.

$$\int d^4x\hat{\delta}_\omega\star(\phi^2_\star)=-\int d^4xM_\omega(\phi^2_\star)=\int d^4x\partial_n(\omega^{mn}x_m\phi^2_\star)=0$$

$$\hat{M}_\omega\star\phi^3_\star=M_\omega(\phi^3_\star)=(M_\omega\phi^2_\star)\star\phi+(\phi^2_\star)\star(M_\omega\phi)$$
$$-\tfrac{i}{2}\omega^p{}_r\vartheta^{rs}\partial_p(\phi^2_\star)\star\partial_s\phi+\tfrac{i}{2}\omega^p{}_r\vartheta^{rs}\partial_s(\phi^2_\star)\star\partial_p\phi$$

Transformation of the kinetic term:

$$\hat{\delta}_\omega(\partial_m\phi\star\partial_n\phi)=-M_\omega(\partial_m\phi\star\partial_n\phi)+\omega_{mr}\partial^r\phi\star\partial_n\phi+\omega_{nr}\partial_m\phi\star\partial^r\phi$$
$$\hat{\delta}_\omega\int d^4x\eta^{mn}\partial_m\phi\star\partial_n\phi=0$$

Nonlocal U(1) gauge symmetry

The nonlocal generalization of the U(1) gauge transformations gives the non-Abelian transformations

$$\delta_aA_m=\partial_ma(x)-i[a,A_m]_\star,\quad\delta\phi=-i[a,\phi]_\star,\quad\delta\chi=-ia\star\chi,\quad\delta\bar{\chi}=i\bar{\chi}\star a$$

$$[\delta_{a_2}, \delta_{a_1}]\chi = -ia_{br} \star \chi, \quad a_{br} = i[a_2, a_1]_\star$$

Gauge covariant derivatives in the nonlocal U(1) gauge theory are

$$\nabla_m \phi = \partial_m \phi + i[A_m, \phi]_\star, \quad \nabla_m \chi = \partial_m \chi + iA_m \star \chi,$$

$$F_{mn} = \partial_m A_n - \partial_n A_m + i[A_m, A_n]_\star, \quad \delta_a F_{mn} = -i[a, F_{mn}]_\star$$

The gauge-invariant nonlocal action is $S_\star = \int d^4x L_\star(A, \phi, \chi)$

$$L_\star = -\tfrac{1}{4} F^{mn} \star F_{mn} + \tfrac{1}{2} \nabla^m \phi \star \nabla_m \phi + \nabla^m \bar{\chi} \star \nabla_m \chi$$

$$\delta_a L_\star = -i[a, L_\star], \quad \int d^4x \delta_a L_\star = -i \int d^4x [a, L_\star] = 0$$

EXERSISE 1

In the operator representation, we should use the following generator

$$\hat{M}_\omega = \frac{1}{2}\omega^{rs}\hat{M}_{rs} = \omega^{rs}\hat{x}_s\hat{\partial}_r - \frac{i}{2}\omega^p{}_r\vartheta^{rs}\hat{\partial}_s\hat{\partial}_p$$

$$\begin{aligned}\hat{M}_\omega \star (\hat{x}^m \star \hat{x}^n) &= \omega^m_r\hat{x}^r \star \hat{x}^n + \omega^n{}_r\hat{x}^r \star \hat{x}^m - \frac{i}{2}\omega^m_r\vartheta^{rn} - \frac{i}{2}\omega^n{}_r\vartheta^{rm}\\ &= \omega^m_r\hat{x}^r \star \hat{x}^n + \omega^n{}_r\hat{x}^m \star \hat{x}^r - \frac{i}{2}\omega^m_r\vartheta^{rn} + \frac{i}{2}\omega^n{}_r\vartheta^{rm}\\ &\neq (\hat{M}_\omega \star \hat{x}^m) \star \hat{x}^n + \hat{x}^m \star (\hat{M}_\omega \star \hat{x}^n)\end{aligned}$$

The alternative derivation of the same transformation:

$$\Delta_t(M_\omega) = \frac{1}{2}\omega^{rs}\Delta_t(M_{mn}) = M_\omega \otimes 1 + 1 \otimes M_\omega$$

$$-\frac{i}{2}\omega^p{}_r\vartheta^{rs}\partial_p \otimes \partial_s + \frac{i}{2}\omega^p{}_r\vartheta^{rs}\partial_s \otimes \partial_p,$$

$$\hat{M}_\omega \star (\hat{x}^m \star \hat{x}^n) = \mu_\star \circ \Delta_t(M_\omega)\hat{x}^m \otimes \hat{x}^n = (\hat{M}_\omega \star \hat{x}^m) \star \hat{x}^n$$

$$+\hat{x}^m \star (\hat{M}_\omega \star \hat{x}^n) - \frac{i}{2}\omega^m_r\vartheta^{rn} + \frac{i}{2}\omega^n{}_r\vartheta^{rm}$$

Antisymmetrization in m and n gives the relation

$$\hat{M}_\omega \star (\hat{x}^m \star \hat{x}^n - \hat{x}^n \star \hat{x}^m) = \omega^m_r[\hat{x}^r, \hat{x}^n]_\star + \omega^n{}_r[\hat{x}^m, \hat{x}^r]_\star - i\omega^m_r\vartheta^{rn} + i\omega^n{}_r\vartheta^{rm}$$

The anticommutator $A^{mn}_\star = \hat{x}^m \star \hat{x}^n + \hat{x}^n \star \hat{x}^m$ is covariant

$$\hat{M}_\omega \star A^{mn}_\star = \omega^m_r A^{rn}_\star + \omega^n{}_r A^{mr}_\star$$

It is not difficult to check a covariance of the basic quantity $C^{mn}_\star = [\hat{x}^m, \hat{x}^n]_\star - i\vartheta^{mn}$ in the operator representation of algebra $A_\star$

$$\hat{M}_\omega \star C^{mn}_\star = \hat{M}_\omega \star (\hat{x}^m \star \hat{x}^n - \hat{x}^n \star \hat{x}^m)$$

$$\begin{aligned}&= \omega^m{}_r[\hat{x}^r, \hat{x}^n]_\star + \omega^n{}_r[\hat{x}^m, \hat{x}^r]_\star - i\omega^m_r\vartheta^{rn} + i\omega^n_r\vartheta^{rm}\\ &= \omega^m{}_r C^{rn}_\star + \omega^n{}_r C^{mr}_\star.\end{aligned}$$

TWIST-DEFORMED N=($\frac{1}{2}$, $\frac{1}{2}$) SUPERSYMMETRY

Let us consider the Euclidean space R^4 and its symmetry group SO(4)$\sim$ SU(2)$_L\times$SU(2)$_R$. The Euclidean Weyl matrices $(\sigma_m)_{\alpha\dot\beta}$ and $(\bar\sigma_m)^{\dot\alpha\beta}$ satisfy the following relations:

$$(\sigma_m\bar\sigma_n + \sigma_n\bar\sigma_m)_\alpha^\beta = 2\delta_{mn}\delta_\alpha^\beta,$$

$$(\bar\sigma_m\sigma_n + \bar\sigma_n\sigma_m)_{\dot\alpha}^{\dot\beta} = 2\delta_{mn}\delta_{\dot\alpha}^{\dot\beta}$$

The spinor coordinates anticommute

$$\{\theta^\alpha, \theta^\beta\} = \{\theta^\alpha, \bar\theta^{\dot\beta}\} = \{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\} = 0$$

and form (2,2) dimensional Grassmann algebra. We use the chiral coordinates $z^M = (y_m, \theta^\alpha, \bar\theta^{\dot\alpha})$ in the Euclidean superspace R$(4|2,2)$, where $m = 1,2,3,4,\quad \alpha = 1,2$, and $\dot\alpha = \dot1, \dot2$. Note that these coordinates are pseudoreal with respect to the special conjugation

$$(y_m)^* = y_m, \quad (\theta^\alpha)^* = \varepsilon_{\alpha\beta}\theta^\beta, \quad (\bar\theta^{\dot\alpha})^* = \varepsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta},$$

so one can use the reality condition for the even Euclidean chiral superfield $\phi(y,\theta)$. The generators G of the Euclidean N=($\frac{1}{2}$, $\frac{1}{2}$) supersymmetry SUSY($\frac{1}{2}$, $\frac{1}{2}$) have the following form:

$$L_\alpha^\beta = L_\alpha^\beta(y) + L_\alpha^\beta(\theta) = \tfrac{1}{4}(\sigma_m\bar\sigma_n)_\alpha^\beta(y_n\partial_m - y_m\partial_n) + \theta^\beta\partial_\alpha - \tfrac{1}{2}\delta_\alpha^\beta\theta^\gamma\partial_\gamma,$$

$$R_{\dot\alpha}^{\dot\beta} = R_{\dot\alpha}^{\dot\beta}(y) + R_{\dot\alpha}^{\dot\beta}(\bar\theta) = \tfrac{1}{4}(\bar\sigma_m\sigma_n)_{\dot\alpha}^{\dot\beta}(y_m\partial_n - y_n\partial_m) + \bar\theta^{\dot\beta}\bar\partial_{\dot\alpha} - \tfrac{1}{2}\delta_{\dot\alpha}^{\dot\beta}\bar\theta^{\dot\gamma}\bar\partial_{\dot\gamma},$$

$$O = \theta^\alpha\partial_\alpha - \bar\theta^{\dot\alpha}\bar\partial_{\dot\alpha}, \quad Q_\alpha = \partial_\alpha, \quad \bar Q_{\dot\alpha} = \bar\partial_{\dot\alpha} - 2i\theta^\alpha\partial_{\alpha\dot\alpha}, \quad P_m = \partial_m$$

where $\partial_M = (\partial_m, \partial_\alpha, \bar\partial_{\dot\alpha})$ are partial derivatives in the chiral coordinates

$$\partial_m y_n = \delta_{mn}, \quad \partial_\alpha\theta^\beta = \delta_\alpha^\beta, \quad \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}.$$

Generators $L_\alpha^\beta, R_{\dot\alpha}^{\dot\beta}$ and O correspond to the automorphism group SU(2)$_L\times$SU(2)$_R\times$O(1,1).

The supersymmetry transformation of the chiral coordinates are

$$\delta y_m = 2i\theta\sigma_m\bar\epsilon, \quad \delta\theta^\alpha = \epsilon^\alpha, \quad \delta\bar\theta^{\dot\alpha} = \bar\epsilon^{\dot\alpha}$$

Thus, the chiral superspace y_m, θ^α is covariant with respect to the supersymmetry. The primary chiral superfield $\phi(y,\theta)$ realizes the irreducible representation of supersymmetry

$$\delta\phi = -(c_m P_m + \omega_\beta^\alpha L_\alpha^\beta + \lambda_{\dot\beta}^{\dot\alpha} R_{\dot\alpha}^{\dot\beta} + aO + \epsilon^\alpha Q_\alpha + \bar\epsilon^{\dot\alpha}\bar Q_{\dot\alpha})\phi$$

Component fields can be introduced via the Grassmann decomposition

$$\phi(y,\theta) = a(y) + \theta^\alpha\psi_\alpha(y) + (\theta^\alpha\theta_\alpha)f(y)$$

using the identities

$$\theta^\alpha\theta^\beta = \tfrac{1}{2}\varepsilon^{\beta\alpha}\theta^2, \quad \theta^\alpha\theta^\beta\theta^\gamma = 0$$

The central superspace coordinates $x_m, \theta^\alpha, \bar\theta^{\dot\alpha}$ contain

$$x_m = y_m - i\theta\sigma_m\bar\theta, \quad \delta x_m = i\theta\sigma_m\bar\epsilon - i\epsilon\sigma_m\bar\theta$$

The right coordinates $\bar y_m, \theta^\alpha, \bar\theta^{\dot\alpha}$ contain

$$\bar y_m = y_m - 2i\theta\sigma_m\bar\theta, \quad \delta\bar y_m = -2i\epsilon\sigma_m\bar\theta$$

The anti-chiral superfield is

$$\bar\phi(\bar y_m, \bar\theta^{\dot\alpha}) = \bar a(\bar y) + \bar\theta^{\dot\alpha}\bar\psi_{\dot\alpha}(\bar y) + \bar\theta^2\bar f(\bar y)$$

The non-anticommutative deformation of the Euclidean $N{=}(\frac{1}{2},\frac{1}{2})$ supersymmetry is based on the following $\star$-product of superfields A and B

$$A\star B = Ae^P B = AB - \tfrac{1}{2}(-1)^{p(A)}C^{\alpha\beta}Q_\alpha A\, Q_\beta B$$

$$-\tfrac{1}{32}C^{\alpha\beta}C_{\alpha\beta}Q^2A\,Q^2B,$$

$$APB = -\tfrac{1}{2}(-1)^{p(A)}C^{\alpha\beta}Q_\alpha A\, Q_\beta B, \quad P^3 = 0$$

where $C^{\alpha\beta}$ are some constants, $p(A)$ is the Z_2 grading, and P is the nilpotent bi-differential operator which contains generators Q_α of the left $N{=}(\frac{1}{2},0)$ supersymmetry. All products of superfields and their derivatives in the right-hand side are supercommutative. We shall use the notation $\mathrm{S}(4|2,2)$ or $\mathrm{C}(4|2,0)$ for the supercommutative algebras of general or chiral superfields and the notation $\mathrm{S}_\star(4|2,2)$ or $\mathrm{C}_\star(4|2,0)$ for the analogous non-anticommutative algebras, respectively. The bilinear multiplication map μ connects the tensor product with the local supercommutative product in $\mathrm{S}(4|2,2)$

$$\mu\circ A\otimes B = AB = (-1)^{p(A)p(B)}BA.$$

The standard coproduct map $\Delta(G) = G\otimes 1 + 1\otimes G$ is defined on the generators of $\mathrm{SUSY}(\frac{1}{2},\frac{1}{2})$. It determines the action of these generators on the tensor product of superfields and yields the standard Leibniz rule for supersymmetry transformations on the local product of superfields $G(AB) = (GA)B + AGB$.

The twist operator in this supersymmetry

$$\mathcal{F} = \exp(\mathcal{P}), \quad \mathcal{P} = -\tfrac{1}{2}C^{\alpha\beta}Q_\alpha\otimes Q_\beta,$$

$$\mathcal{F}(A\otimes B) = A\otimes B - \tfrac{1}{2}(-1)^{p(A)}C^{\alpha\beta}Q_\alpha A\otimes Q_\beta B$$
$$-\tfrac{1}{32}C^{\alpha\beta}C_{\alpha\beta}Q^2A\otimes Q^2B.$$

The bilinear map $\mu_\star$ in $\mathrm{S}_\star(4|2,2)$ can be defined via this twist operator

$$A\star B \equiv \mu_\star\circ A\otimes B = \mu\circ\exp(\mathcal{P})A\otimes B,$$

so $\mathcal{F}$ is the quantum group analog of the pseudolocal operator e^P.

By analogy with the map between differential operators on commutative and noncommutative algebras of functions , we can easily define the corresponding differential operator $\hat X_D$ on $\mathrm{S}_\star(4|2,2)$ for any differential operator D on the supercommutative algebra. In the case of the

1-st order operator $D = \xi^M(z)\partial_M$, the image $\hat{X}_D$ contains, in general, terms with an arbitrary number of derivatives on $\mathrm{S}_\star(4|2,2)$

$$(\hat{\partial}_M \star A) = \partial_M A, \quad (\hat{\partial}_M \star z^N) = \delta_M^N,$$

$$(\hat{X}_D \star A) = (D_1 A) = \mu_\star \circ \exp(-\mathcal{P})(\xi^M(z) \otimes \partial_M)(1 \otimes A)$$
$$= \xi^M(z) \star \partial_M A + \tfrac{1}{2}(-1)^{p(D)} C^{\alpha\beta} Q_\alpha \xi^M(z) \star \partial_M Q_\beta A + O(C^2)$$

where $p(D)$ is the Z_2 grading of D. For instance, the non-anticommutative images of generators $\bar{Q}_{\dot{\alpha}}$ and L_a^β are the following second-order differential operators on $\mathrm{S}_\star(4|2,2)$:

$$(\hat{\bar{Q}}_{\dot{\alpha}} \star A) = (\bar{\partial}_{\dot{\alpha}} - 2i\theta^\alpha \partial_{\alpha\dot{\alpha}} + iC^{\alpha\beta}\partial_{\alpha\dot{\alpha}} Q_\beta) \star A = \bar{Q}_{\dot{\alpha}} A,$$

$$(\hat{L}_\alpha^\beta \star A) = (L_\alpha^\beta - \tfrac{1}{2} C^{\beta\gamma} Q_\gamma Q_\alpha) \star A = L_\alpha^\beta A,$$

while $\hat{R}_{\dot{\alpha}}^{\dot{\beta}} = R_{\dot{\alpha}}^{\dot{\beta}}$ and $\hat{O} = O$.

The generators with hats have the same Lie supercommutators as the standard supersymmetry generators of $\mathrm{SUSY}(\frac{1}{2}, \frac{1}{2})$, so it seems convenient to use the 1-st order differential operators G as the basic generators of the twist-deformed quantum supergroup $\mathrm{SUSY}_t(\frac{1}{2}, \frac{1}{2})$, too.

The coproduct $\Delta_t(G) = e^{-\mathcal{P}}\Delta(G)e^{\mathcal{P}}$ in $\mathrm{SUSY}_t(\frac{1}{2}, \frac{1}{2})$ is deformed on the part of generators

$$\Delta_t(\bar{Q}_{\dot{\alpha}}) = \bar{Q}_{\dot{\alpha}} \otimes 1 + 1 \otimes \bar{Q}_{\dot{\alpha}} + iC^{\alpha\beta}(\partial_{\alpha\dot{\alpha}} \otimes Q_\beta - Q_\alpha \otimes \partial_{\beta\dot{\alpha}}),$$

$$\Delta_t(L_\alpha^\beta) = L_\alpha^\beta \otimes 1 + 1 \otimes L_\alpha^\beta + \tfrac{1}{2} C^{\rho\sigma}(\delta_\rho^\beta Q_\alpha \otimes Q_\sigma$$
$$+\delta_s^\beta Q_\rho \otimes Q_\alpha - \delta_\alpha^\beta Q_\rho \otimes Q_\sigma),$$
$$\Delta_t(O) = O \otimes 1 + 1 \otimes O - C^{\alpha\beta} Q_\alpha \otimes Q_\beta$$

The coproduct on the generators $R_{\dot{\alpha}}^{\dot{\beta}}, P_m, Q_\alpha$ is not deformed.

Acting by the composition of $\mu_\star$ and coproduct $\Delta_t(\bar{Q}_{\dot{\alpha}})$ on the tensor product of superfields, one can obtain the deformed Leibniz rule for operator $\hat{\bar{Q}}_{\dot{\alpha}}$ on the $\star$-product

$$\mu_\star \circ \Delta_t(\bar{Q}_{\dot{\alpha}}) A \otimes B \equiv \hat{\bar{Q}}_{\dot{\alpha}} \star (A \star B) = (\bar{Q}_{\dot{\alpha}} A) \star B + (-1)^{p(A)} A \star \bar{Q}_{\dot{\alpha}} B$$
$$+iC^{\alpha\beta}[(-1)^{p(A)} \partial_{\alpha\dot{\alpha}} A \star Q_\beta B - Q_\alpha A \star \partial_{\beta\dot{\alpha}} B]$$

It is evident that the algebra $\mathrm{S}_\star(4|2,2)$ is not covariant with respect to the undeformed supersymmetry, e.g., $\delta_{\bar{\epsilon}}(A \star B) \neq \bar{\epsilon}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}(A \star B)$. At the field-theory level, it is important to formulate the principle of covariance of the algebra $\mathrm{S}_\star(4|2,2)$ (or $\mathrm{C}_\star(4|2,0)$) with respect to all transformations of $\mathrm{SUSY}_t(\frac{1}{2}, \frac{1}{2})$: the superfields $(A \star B), A$ and B transforms analogously

$$\hat{G} \star (A \star B) = G(A \star B), \quad \hat{G} \star A = GA, \quad \hat{G} \star B = GB,$$

where the manifest representation for generators G should be used. Note that the quantity AB is not covariant in this sense. For instance, it is not difficult to define the noncovariant action of generators $\hat{\bar{Q}}_{\dot{\alpha}}$ and $\hat{L}_\alpha^\beta$ on the ordinary product of the even chiral superfields

$$\hat{\bar{Q}}_{\dot{\alpha}} \star (\phi_1 \phi_2) \equiv \mu \circ \Delta_t(\bar{Q}_{\dot{\alpha}}) \phi_1 \otimes \phi_2 = \bar{Q}_{\dot{\alpha}}(\phi_1 \phi_2) + iC^{\alpha\beta}(\partial_{\alpha\dot{\alpha}} \phi_1 \, Q_\beta \phi_2$$

$$-Q_\alpha\phi_1\,\partial_{\beta\dot\alpha}\phi_2),$$

$$\hat{L}^\beta_\alpha\star(\phi_1\phi_2)\equiv\mu\circ\Delta_t(L^\beta_\alpha)\phi_1\otimes\phi_2=L^\beta_\alpha(\phi_1\phi_2)$$

$$+\tfrac{1}{2}C^{\rho\sigma}(\delta^\beta_\rho Q_\alpha\phi_1 Q_\sigma\phi_2+\delta^\beta_\sigma Q_\rho\phi_1 Q_\alpha\phi_2-\delta^\beta_\alpha Q_\rho\phi_1 Q_\sigma\phi_2)$$

The first terms coincide with the transformations of the undeformed supersymmetry. Using a θ-decomposition of these superfield formulas one can obtain the deformed transformations of the products of component fields, for instance,

$$\hat{\bar{Q}}_{\dot\alpha}(a_1a_2)=iC^{\alpha\beta}(\partial_{\alpha\dot\alpha}a_1\,\psi_{\beta2}-\psi_{\alpha1}\partial_{\beta\dot\alpha}a_2),$$

$$\hat{L}^\beta_\alpha(a_1a_2)=L^\beta_\alpha(y)(a_1a_2)+\tfrac{1}{2}C^{\rho\sigma}(\delta^\beta_\rho\psi_{\alpha1}\psi_{\sigma2}+\delta^\beta_\sigma\psi_{\rho1}\psi_{\alpha2}-\delta^\beta_\alpha\psi_{\rho1}\psi_{\sigma2})$$

Let us consider two even chiral superfields ϕ_1 and ϕ_2 in the chiral basis

$$\phi_i=a_i+\theta^\alpha\psi_{i\alpha}+\theta^2f_i,$$

$$Q_\alpha\phi_i=\psi_{i\alpha}+2\theta_\alpha f_i,\quad Q^2\phi_i=-4f_i$$

The θ-decomposition of the $\star$-product of two chiral superfields depends on these components and constants $C^{\alpha\beta}$

$$\Phi_{12}=\phi_1\star\phi_2=\mathcal{B}+\theta^\alpha\Psi_\alpha+\theta^2F,$$

$$\mathcal{B}=a_1a_2-\tfrac{1}{2}C^{\alpha\beta}\psi_{1\alpha}\psi_{2\beta}-\tfrac{1}{2}C^{\alpha\beta}C_{\alpha\beta}f_1f_2,$$

$$\Psi_\alpha=a_1\psi_{2\alpha}+a_2\psi_{1\alpha}-C_{\alpha\beta}(f_1\psi^\beta_2-f_2\psi^\beta_1),$$

$$F=a_1f_2+a_2f_1-\tfrac{1}{2}\psi^\alpha_1\psi_{2\alpha}$$

These relations can be treated as a deformed tensor calculus for chiral component multiplets. The t-supersymmetry transformations of the composite components are completely analogous to the transformations of the basic components $a_i,\psi_{\alpha i},f_i$

$$\hat{\bar{Q}}_{\dot\alpha}\mathcal{B}=0,\quad\hat{\bar{Q}}_{\dot\alpha}\Psi_\alpha=2i\partial_{\alpha\dot\alpha}\mathcal{B},\quad\hat{\bar{Q}}_{\dot\alpha}F=i\partial_{\alpha\dot\alpha}\Psi^\alpha,$$

$$\hat{L}^\beta_\alpha\mathcal{B}=L^\beta_\alpha(y)\mathcal{B},\quad\hat{L}^\beta_\alpha\Psi_\gamma=-\delta^\beta_\gamma\Psi_\alpha+(L^\beta_\alpha(y)+\tfrac{1}{2}\delta^\beta_\alpha)\Psi_\gamma,\quad\hat{L}^\beta_\alpha F=L^\beta_\alpha(y)F$$

These transformations are compatible with the noncovariant component transformations.

The non-anticommutative deformation of the Euclidean model for an arbitrary number of the chiral and antichiral superfields ϕ_a and $\bar\phi_a$ is based on the superfield action $S_\star(\phi_a,\bar\phi_a)$. Each term of the $\star$-polynomial decomposition of this action is separately invariant with respect to $\mathrm{SUSY}_t(\frac{1}{2},\frac{1}{2})$, while the quadratic terms like $\int d^8z\,\phi_a\star\bar\phi_a=\int d^8z\,\phi_a\bar\phi_a$ possess also the undeformed supersymmetry.

TWIST-DEFORMED $N{=}(1,1)$ SUPERSYMMETRY

The nilpotent deformations of the Euclidean $N{=}(1,1)$ supersymmetry were considered in the framework of the harmonic-superspace approach. We shall use the SU(2)/U(1) harmonics $u^\pm_i$ and the chiral superspace coordinates

$$z^M = (y_m, \theta^\alpha_k, \bar\theta^{\dot\alpha k})\,,\quad y_m = x_m + i\theta_k\sigma_m\bar\theta^k$$

where x_m are the central 4D coordinates. The spinor derivatives $D^k_\alpha, \bar D_{\dot\alpha k}$ in these coordinates are

$$D^k_\alpha = \partial^k_\alpha + 2i\bar\theta^{\dot\alpha k}\partial_{\alpha\dot\alpha},\quad \bar D_{\dot\alpha k} = \bar\partial_{\dot\alpha k}.$$

Using harmonic projections of the Grassmann coordinates $\theta^{\pm\alpha} = u^\pm_k\theta^{\alpha k}$ and $\bar\theta^{\pm\dot\alpha} = u^\pm_k\bar\theta^{\dot\alpha k}$ one can define the analytic coordinates $(x_A, \theta^\pm, \bar\theta^\pm)$. The bosonic part of the harmonic superspaces is $R^4\times S^2$, and the left-right Grassmann dimensions of general, chiral or G-analytic superspaces are (4,4), (4,0) or (2,2), respectively. We shall use the notation $\mathrm{S}(4,2|4,4)$ for the supercommutative algebra and $\mathrm{S}_\star(4,2|4,4)$ for the non-anticommutative algebra of general harmonic superfields, respectively.

It is convenient to use the following differential representation of the SUSY(1,1) generators $\mathcal{G}$ on $\mathrm{S}(4,2|4,4)$:

$$T^k_l = -\theta^\alpha_l\partial^k_\alpha + \tfrac12\delta^k_l\theta^\alpha_j\partial^j_\alpha + \bar\theta^{\dot\alpha k}\bar\partial_{\dot\alpha l} - \tfrac12\delta^k_l\bar\theta^{\dot\alpha j}\bar\partial_{\dot\alpha j} - u^\pm_l\partial^{\mp k} + \tfrac12\delta^k_l u^\pm_j\partial^{\mp j},$$

$$L^\beta_\alpha = L^\beta_\alpha(y) + \theta^\beta_k\partial^k_\alpha - \tfrac12\delta^\beta_\alpha\theta^\gamma_k\partial^k_\gamma,\quad R^{\dot\beta}_{\dot\alpha} = R^{\dot\beta}_{\dot\alpha}(y) + \bar\theta^{\dot\beta k}\bar\partial_{\dot\alpha k} - \tfrac12\delta^{\dot\beta}_{\dot\alpha}\bar\theta^{\dot\gamma k}\bar\partial_{\dot\gamma k},$$

$$O = \theta^\alpha_k\partial^k_\alpha - \bar\theta^{\dot\alpha k}\bar\partial_{\dot\alpha k},\quad Q^k_\alpha = \partial^k_\alpha,\quad \bar Q_{\dot\alpha k} = \bar\partial_{\dot\alpha k} - 2i\theta^\alpha_k\partial_{\alpha\dot\alpha},\quad P_m = \partial_m$$

where $L^\beta_\alpha(y)$ and $R^{\dot\beta}_{\dot\alpha}(y)$ are defined above , and partial harmonic derivatives act as follows $\partial^{\mp l}u^\pm_k = \delta^l_k$.

The $N{=}(1,1)$ twist operator $\mathcal{F} = \exp(\mathcal{P})$ contains the nilpotent operator

$$\mathcal{P} = -\tfrac12 C^{\alpha\beta}_{kl}Q^k_\alpha\otimes Q^l_\beta,\quad \mathcal{P}^5 = 0$$

where $C^{\alpha\beta}_{kl}$ are some constants. The non-anticommutative product in the corresponding deformed algebra $\mathrm{S}_\star(4,2|4,4)$ can be defined by equivalent formulas

$$A\star B = A\exp(P)B = \mu\circ\exp(\mathcal{P})A\otimes B = \mu_\star\circ A\otimes B$$

where μ and $\mu_\star$ are product maps for $\mathrm{S}(4,2|4,4)$ and $\mathrm{S}_\star(4,2|4,4)$ and P is the basic operator

$$APB = -\tfrac12(-1)^{p(A)}C^{\alpha\beta}_{kl}Q^k_\alpha A\, Q^l_\beta B = \mu\circ\mathcal{P}\,A\otimes B.$$

The non-anticommutative algebras of the $N{=}(1,1)$ chiral or G-analytic superfields can be defined analogously.

One can construct the image operator $\hat X_{D_1}$ on $\mathrm{S}_\star(4,2|4,4)$ for any 1-st order differential operator $D_1 = \xi^M(z)\partial_M$ on $\mathrm{S}(4,2|4,4)$. In the twist-deformed $N{=}(1,1)$ supergroup SUSY$_t$(1,1), we prefer to use the standard 1-st order representation of generators , although the equivalent higher-order representation of generators on $\mathrm{S}_\star(4,2|4,4)$ can be easily constructed.

The coproduct of SUSY$_t$(1,1) $\Delta_t(\mathcal{G})=e^{-\mathcal{P}}\Delta(\mathcal{G})e^{\mathcal{P}}$ is deformed on the following generators:

$$\Delta_t(\bar{Q}_{\dot{\alpha}k})=\bar{Q}_{\dot{\alpha}k}\otimes 1+1\otimes\bar{Q}_{\dot{\alpha}k}+iC^{\alpha\beta}_{kj}\partial_{\alpha\dot{\alpha}}\otimes Q^j_\beta-iC^{\alpha\beta}_{ik}Q^i_\alpha\otimes\partial_{\beta\dot{\alpha}},$$

$$\begin{aligned}\Delta_t(T^k_l)&=T^k_l\otimes 1+1\otimes T^k_l-\tfrac{1}{2}C^{\alpha\beta}_{lj}Q^k_\alpha\otimes Q^j_\beta\\&-\tfrac{1}{2}C^{\alpha\beta}_{jl}Q^j_\alpha\otimes Q^k_\beta+\tfrac{1}{2}\delta^k_lC^{\alpha\beta}_{ij}Q^i_\alpha\otimes Q^j_\beta,\\\Delta_t(L^\beta_\alpha)&=L^\beta_\alpha\otimes 1+1\otimes L^\beta_\alpha+\tfrac{1}{2}C^{\beta\gamma}_{kl}Q^k_\alpha\otimes Q^l_\gamma\\&+\tfrac{1}{2}C^{\rho\beta}_{kl}Q^k_\rho\otimes Q^l_\alpha-\tfrac{1}{2}\delta^\beta_\alpha C^{\rho\gamma}_{kl}Q^k_\rho\otimes Q^l_\gamma,\\\Delta_t(O)&=O\otimes 1+1\otimes O-C^{\alpha\beta}_{kl}Q^k_\alpha\otimes Q^l_\beta\end{aligned}$$

and it is not deformed on $P_m, R^{\dot{\beta}}_{\dot{\alpha}}$ and Q_α.

The $\star$-products of arbitrary N=$(1,1)$ superfields preserve covariance with respect to all transformations of SUSY$_t$(1,1)

$$\widehat{\mathcal{G}}\star(A\star B)\equiv\mu_\star\circ\Delta_t(\mathcal{G})A\otimes B=\mathcal{G}(A\star B).$$

The deformed Leibniz rules can be derived from these covariant relations.

The deformed transformations act noncovariantly on the supercommutative product of superfields, for instance,

$$\begin{aligned}&\widehat{\bar{Q}}_{\dot{\alpha}k}\star(AB)\equiv\mu\circ\Delta_t(\bar{Q}_{\dot{\alpha}k})A\otimes B\\&=\bar{Q}_{\dot{\alpha}k}(AB)+iC^{\alpha\beta}_{kj}(-1)^{p(A)}\partial_{\alpha\dot{\alpha}}A\,Q^j_\beta B-i\bar{\epsilon}^{\dot{\alpha}k}C^{\alpha\beta}_{ik}Q^i_\alpha A\,\partial_{\beta\dot{\alpha}}B.\end{aligned}$$

It is easy to define the t-deformed transformations on the products of the N=$(1,1)$ component fields using the corresponding Grassmann decompositions.

In the special case of the singlet deformation , the twist operator is defined by the parameter I and the SU(2)$\times$SU(2)$_L$ invariant constant tensor

$$C^{\alpha\beta}_{kl}=2I\varepsilon^{\alpha\beta}\varepsilon_{kl}\ \Rightarrow\ \mathcal{P}_s=-IQ^i_\alpha\otimes Q^\alpha_i.$$

The $\mathcal{P}_s$-twist deformation vanishes for SU(2) and SU(2)$_L$ transformations.

The Leibniz rules for differential operators $D=(\partial_m,D^k_\alpha,\bar{D}_{\dot{\alpha}k},\partial/\partial u^\pm_k)$ are standard for the general Q-deformation

$$D(A\star B)=(DA\star B)+(-1)^{p(D)p(A)}(A\star DB).$$

The $\star$-product preserves differential constraints of chirality, anti-chirality and Grassmann analyticity. All superfield actions using $\star$-products in the non-anticommutative $N=(1,1)$ harmonic superspace are invariant with respect to the quantum group SUSY$_t$(1,1), and this invariance is a natural basic principle of these deformed theories. Free quadratic parts of these actions possess also the undeformed $N=(1,1)$ supersymmetry. The simple examples of the superfield-density terms for the G-analytic hypermultiplet q^+, $\tilde{q}^+$ and the U(1) gauge potential V^{++} in the deformed theory are

$$\tilde{q}^+\star(D^{++}q^++[V^{++},q^+]_\star)+\lambda q^+\star q^+\star\tilde{q}^+\star\tilde{q}^+.$$

The t-supersymmetry transformations of any term $L^{+4}_\star$ in this density are

$$(\hat{\delta}_{\bar\epsilon} + \hat{\delta}_u + \hat{\delta}_l) \star L^{+4}_\star = (\bar\epsilon^{\dot\alpha k}\bar{Q}_{\dot\alpha k} + u^l_k T^k_l + l^\alpha_\beta L^\beta_a) L^{+4}_\star ,$$

so the analytic-superspace integrals of these variations vanish. We hope that the manifest $\mathrm{SUSY}_t(1,1)$ covariance could help to prove the nonrenormalization theorems in t-deformed harmonic-superfield theories by analogy with the corresponding undeformed theories.

The singlet D-deformation of the $N{=}(1,1)$ supersymmetry corresponds to the alternative singlet twist operator constructed in terms of spinor derivatives D^k_α

$$(A \star B)_D = \mu \circ \exp(\mathcal{P}_D) A \otimes B, \quad \mathcal{P}_D = -J D^k_\alpha \otimes D^\alpha_k ,$$

where J is some constant. Now the Leibniz rules are deformed for $\bar{D}_{\dot\alpha k}$ and the O(1,1) generator and undeformed for all other generators of SUSY(1,1).

The singlet D-twist does not deform products of anti-chiral or G-analytic superfields. The D-deformation is interesting for the general-superfield objects of the superfield geometry in the $N{=}(1,1)$ gauge theory.

References

[1] M.R. Douglas, N.A. Nekrasov, Rev. Mod. Phys. **73** (2001) 977, hep-th/0106048; R.J. Szabo, Phys. Rep. **378** (2003) 207, hep-th/0109162.

[2] N. Seiberg and E. Witten, JHEP **9909** (1999) 032 [hep-th/9908142].

[3] V.G. Drinfeld, Leningrad Math. J. **1** (1990) 1419.

[4] R. Oeckl, Nucl. Phys. B 581 (2000) 559, hep-th/0003018.

[5] M. Chaichian, P.P. Kulish, K. Nishijima, A. Tureanu, Phys. Lett. **B 604** (2004) 98, hep-th/0408069.

[6] M. Chaichian, P. Presnajder, A. Tureanu, Phys. Rev. Lett. **94** (2005) 151602, hep-th/0409096.

[7] J. Wess, Deformed coordinate spaces. Derivatives, hep-th/0408080.

[8] P. Ashieri, C. Blohmann, M. Dimitrijević, F. Meyer, P. Schupp, J. Wess, A gravity theory on noncommutative spaces, hep-th/0504183.

[9] P. Kosiński, P. Maślanka, Lorentz-invariant interpretation of noncommutative space-time – global version, hep-th/0408100.

[10] S. Ferrara and M.A. Lledó, JHEP **0005** (2000) 008 [hep-th/0002084].

[11] D. Klemm, S. Penati and L. Tamassia, Class. Quant. Grav. **20** (2003) 2905 [hep-th/0104190].

[12] S. Ferrara, M.A. Lledó and O. Maciá, JHEP **0309** (2003) 068 [hep-th/0307039].

[13] N. Seiberg, JHEP **0306** (2003) 010, hep-th/0305248.

[14] E. Ivanov, O. Lechtenfeld, B. Zupnik, JHEP **0402** (2004) 012, hep-th/0308012.

[15] S. Ferrara, E. Sokatchev, Phys. Lett. **B 579** (2004) 226, hep-th/0308021.

[16] A.T. Banin, I.L. Buchbinder, N.G. Pletnev, JHEP **0407** (2004) 011, hep-th/0405063.

[17] S. Ferrara, E. Ivanov, O. Lechtenfeld, E. Sokatchev, B. Zupnik, Nucl. Phys. **B704** (2005) 154, hep-th/0405049.

[18] E. Ivanov, O. Lechtenfeld, B. Zupnik, Nucl. Phys. **B 707** (2005) 69, hep-th/0408146.

[19] Y. Kobayashi, S. Sasaki, Lorentz invariant and supersymmetric interpretation of noncommutative quantum field theory, hep-th/0410164.

[20] B.M. Zupnik, Twist-deformed supersymmetries in non-anticommutative superspace, hep-th/0506043.

[21] M. Ihl, C. Sämann, Drinfeld-twisted supersymmetry and non-anticommutative superspace, hep-th/0506057.