Quantum transport. Self-consistent approximations.

D.N. Voskresensky

MEPhI, Moscow & GSI, Darmstadt

{*results of collaboration with Yu.B. Ivanov (KIAE Moscow and & GSI, Darmstadt) and J. Knoll, GSI, Darmstadt*}

✔ Functional methods: Generating $\Phi\{\phi, G\}$ functional of Baym on Schwinger-Keldysh contour; symmetries and conservation laws; thermodynamic consistency

Yu.B. Ivanov, J. Knoll, D.V.: Nucl. Phys. **A657** (1999) 413.

Renormalization scheme can be found in H. van Hees, J. Knoll: Phys. Rev. **D65** (2002) 025010; 105005; **D66** (2002) 025028.

✔ Generalized kinetic approach: gradient approximation

Yu.B. Ivanov, J. Knoll, D.V.: Nucl. Phys. **A672** (2000) 313; Ann. Phys. **293** (2001) 126; Phys. At. Nucl. **66** (2003) 1902.

Gradient expansion diagram tech. is developed in J. Knoll, Yu.B. Ivanov, D.V.: Ann. Phys. **293** (2001) 126.

✔ Some examples:

Yu.B. Ivanov, J. Knoll, H. van Hees, D.V.: Phys. At. Nucl. **64** (2001) 652; D.V.: Nucl. Phys. **A744** (2004) 378.

Non-equilibrium system of strongly interacting particles

We consider strongly interacting dynamical and (quasi)equilibrium systems of off-mass-shell particles (resonances).

As a particular example, meson-baryon and quark-gluon dynamics in h.i.c.

New facilities: JINR nuclotron, FAIR GSI, RHIC, LHC.

Another example: Neutron stars, in particular neutrino transport in neutron stars and supernovas.

Many examples in condensed matter physics; dynamical conditions in laser fields, etc.

- Perturbative methods are invalid.

One needs resummation schemes described in terms of Green functions and self-energies with particle widths $\Gamma = -2\mathrm{Im}\Pi^R \neq 0$. E.g., for Δ, ρ width $\Gamma \sim 100$ MeV

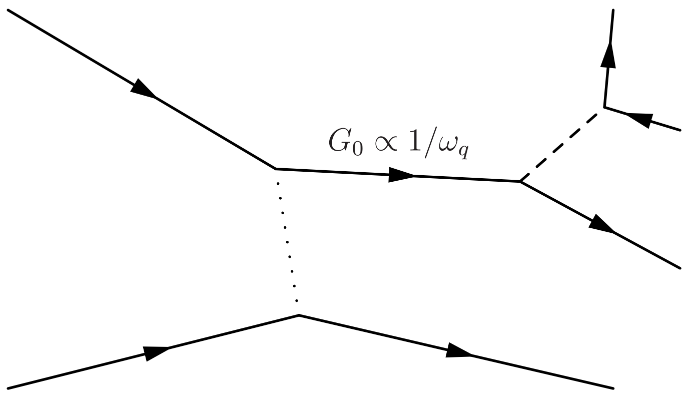
- More, one needs a consistent approximate description scheme

where one should keep conservation laws (better exact or at least approximate).

We will focus on meson-baryon and quark-gluon dynamics in h.i.c.

Example: width effects in infrared radiation problem.

J.Knoll, D.V.: Ann. Phys. **249** (1996) 532

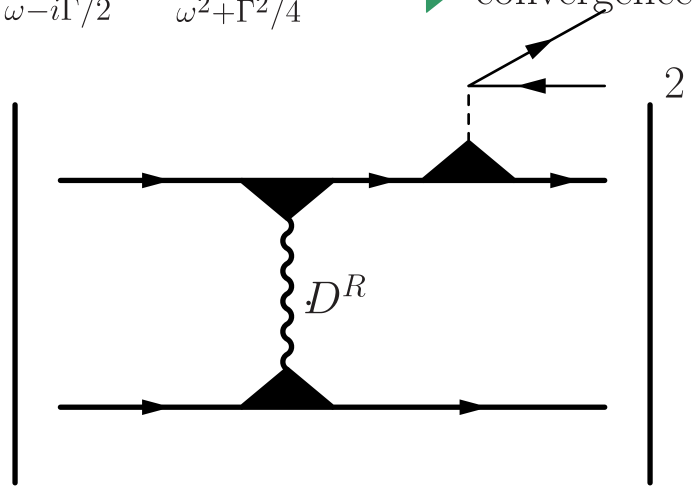


Dash line - in-going/out-going quantum, e.g., photon. In dilepton case virtual quantum produces $l_1 l_2$ dileptons, e.g., ee^+ in h.i.c. or $\bar{\nu}e$, $\bar{\nu}\nu$ in neutron star interiors.

For small ω_q the number of radiated quanta N_ω diverges.

- **A simple idea:** to replace matrix element to that with full interaction and full Green functions.

Then $G_0 \to G \simeq \frac{1}{\omega - i\Gamma/2} = \frac{\omega + i\Gamma/2}{\omega^2 + \Gamma^2/4}$ ➡ convergence:

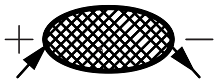


Example: width effects in infrared radiation problem.

But: External legs in matrix elements should be free (otherwise double counting due to possibility of decays from legs). $\Rightarrow$ Closed non-equilibrium diagram technique is welcome $\Rightarrow$ a particular role of $-i\Pi^{-+}$ self-energy in non-eq. Green func. technique:

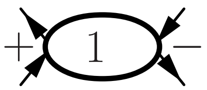
D.V., A.V.Senatorov: Sov. J. Nucl.Phys. **45** (1987) 657; J.Knoll, D.V.: Ann. Phys. **249** (1996) 532

$$\frac{dW}{d^3q/[(2\pi)^3 2\omega_q]} = -i\Pi^{-+} =$$

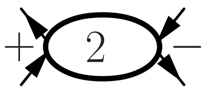


Various contributions can be classified according to the number $N = 1, 2$, etc, of G^{-+} non-eq. Green func. lines (or $G^{-+}G^{+-}$ loops) in the diagram. For $T \ll \epsilon_F$ each G^{-+} brings $(T/\epsilon_F)^2$ small factor. $\Rightarrow$ cut expansion on certain number of loops.

$$\frac{d\mathcal{W}^{\rm tot}_{l_1 l_2}}{dt} = \frac{d^3 q_{l_1} d^3 q_{l_2}}{(2\pi)^6\, 4\,\omega_{l_1}\,\omega_{l_2}} \Big($$



$$+$$



$$\ldots \Big)$$

with full Green functions. $\Rightarrow$ Already first diagram yields convergent number of infrared quanta. But all of diagrams yield rates of the same order for $\omega_q \to 0$. $\Rightarrow$

in infrared limit infinite series should be summed up! How to construct a general approx. theory (or a consistent effective theory)?

Φ derivable methods

- **Conservation laws:**

We want to construct approx. theory with **exactly fulfilled conservation laws.**

Very important for dynamical simulations.

If we do not succeed with such a theory:

- (i) at least conservation laws should be kept approximately,
- (ii) let us construct an effective theory with such properties.

➡ a Φ derivable scheme:

series in terms of full Green functions and bare vertices. Cutting of series on any number of diagrams does not spoil symmetries and self-consistency! ➡ appropriate effective theory.,

as pioneered by G.Baym: Phys. Rev. **127** (1962) 1391

Lagrangian

$$\widehat{L}^0 = -\frac{1}{2}\left(i\partial_t \widehat{\phi}_a^\dagger \widehat{\phi}_a - i\widehat{\phi}_a^\dagger \partial_t \widehat{\phi}_a + m_a^{-1}\nabla\widehat{\phi}_a^\dagger \nabla\widehat{\phi}_a\right) \quad \text{for non-rel. part.}$$

$$\widehat{L}^0 = \frac{1}{2}\left(\partial_\mu \widehat{\phi}_a \partial^\mu \widehat{\phi}_a - m_a^2 (\widehat{\phi}_a)^2\right) \quad \text{for spinless neutral rel. part.}$$

$$\widehat{L}^0 = \partial_\mu \widehat{\phi}_a^\dagger \partial^\mu \widehat{\phi}_a - m_a^2 \widehat{\phi}_a^\dagger \widehat{\phi}_a \quad \text{for spinless charged rel. part.}$$

charge e_a: electric, baryonic, strangeness, etc.

a is particle index (we will often suppress it for briefness).

$$\widehat{L} = \widehat{L}^0 + \widehat{L}^{int}.$$

To simplify notations let us deal with non-derivative coupling theories.

Generalization see in Ivanov, Knoll, D.V., Phys. At. Nucl. **66** (2003) 1902.

Eqs of motion:

$$S_x^a \widehat{\phi}_a(x) = -\widehat{J}_a(x) = -\delta \widehat{L}^{int}/\delta\widehat{\phi}_a^\dagger(x), \quad x = x^\mu = (t, \boldsymbol{r}),$$

$\widehat{J}_a$ is a local source current of field $\widehat{\phi}$.S_x is different. operator of free evolution:

$S_x^a = i\partial_t + \frac{1}{2m_a}\nabla^2$, for non-rel. part., $S_x^a = -\partial_\mu\partial^\mu - m_a^2$ for rel. bosons.

Symmetries and Noether 4-current

•**Noether 4-Current**

$U(1)$ symmetry of the Lagrangian:

$$\widehat{\phi}_a(x) \longrightarrow e^{(-ie_a\Lambda(x))}\widehat{\phi}_a(x) \quad \Rightarrow \quad \text{conserved Noether current}$$

$$\widehat{j}^{\mu}_{Noether} = i\sum_a e_a(\widehat{\phi}^{\dagger}_a \widehat{\pi}^{\mu}_x \widehat{\phi}_a + \widehat{\pi}^{*\mu}_x \widehat{\phi}^{\dagger}_a \cdot \widehat{\phi}_a)$$

$$\widehat{\pi}^{\mu}_x = \tfrac{1}{2}(1, -i\nabla/m_a) \quad \text{for non-rel. part.;} \quad \widehat{\pi}^{\mu}_x = \widehat{p}^{\mu}_x = i\partial^{\mu}_x \quad \text{for rel. bosons.}$$

$$\partial_\mu \widehat{j}^{\mu}_{Noether} = 0$$

Symmetries and Noether energy-momentum tensor

•Noether Energy-Momentum Tensor

Space-time invariance: ➡

$$\widehat{\theta}^{\mu\nu}_{canon}(x) = \sum_a^* \left(\frac{\partial \widehat{L}}{\partial \partial_\mu \widehat{\phi}_a} \partial^\nu \widehat{\phi}_a + \partial^\nu \widehat{\phi}^\dagger_a \cdot \frac{\partial \widehat{L}}{\partial \partial_\mu \widehat{\phi}^\dagger_a} \right) - g^{\mu\nu} \widehat{L}.$$

Notation $\sum_a^*$ includes 1/2 for neutral rel. bosons.

Symmetric (alternative) form:

$$\widehat{\theta}^{\mu\nu}_{alt}(x) = \frac{1}{2} \sum_a^* \left[(\widehat{p}^{*\nu}_x + \widehat{p}^\nu_x)(\widehat{\pi}^{*\mu}_x + \widehat{\pi}^\mu_x) \widehat{\phi}^\dagger_a(x) \widehat{\phi}^\dagger_a(y) \right]_{x=y} + g^{\mu\nu} \left(\widehat{\epsilon}^{int}(x) - \widehat{\epsilon}^{pot}(x) \right)$$

$$\widehat{\epsilon}^{int}(x) = -L^{int}(x), \quad \widehat{\epsilon}^{pot}(x) = -\frac{1}{2} \sum_a^* \left(\widehat{J}^\dagger_a \widehat{\phi}_a + \widehat{\phi}^\dagger_a \widehat{J}_a \right)$$

$$\partial_\mu \widehat{\theta}^{\mu\nu}_{alt} = \partial_\mu \widehat{\theta}^{\mu\nu}_{canon} = 0$$

For interaction with certain number α of operators attached to vertex: $\widehat{\epsilon}^{int} = \frac{2}{\alpha} \widehat{\epsilon}^{pot}$.

Examples: $\lambda \phi^4/4! \rightarrow \alpha = 4$; $\quad \widehat{\phi}^\dagger_f \widehat{\phi}_f (\widehat{\phi}_b + \widehat{\phi}^\dagger_b) : \widehat{\epsilon}^{int} = \frac{2}{\alpha_f + \alpha_b} (\widehat{\epsilon}^{pot}_f + \widehat{\epsilon}^{pot}_b), \alpha_f = 2, \alpha_b = 1$

Non-equilibrium diagram technique. Matrix presentation

see E.Lifshiz, L. Pitaevsky Phys. Kinetics, Pergamon 1981.

Ground state:

$$iG(x,y) = <0|\widehat{T}\widehat{\phi}(x)\widehat{\phi}^{\dagger}(y)|0> = <0|\widehat{S}^{-1}\widehat{T}\widehat{\phi}_I(x)\widehat{\phi}_I^{\dagger}(y)|\widehat{S}|0>$$

in interact. picture.

$$iG = <0|\widehat{T}\widehat{\phi}_I(x)\widehat{\phi}_I^{\dagger}(y)|\widehat{S}|0> / \widehat{S}$$

since ground state transforms to ground state under action of evolution operator

$$\widehat{S} = \widehat{T}e^{(-i\int_{-\infty}^{\infty}\widehat{V}_I(t)dt)}, \quad \widehat{V}_I = e^{i\widehat{H}_I(\mu)t}\widehat{V}(t) = e^{-i\widehat{H}_I(\mu)t},$$

$\widehat{T}$ is time ordering, $\widehat{H}(\mu) = H - \sum_a \mu_a \widehat{N}_a$, $\mu_a \neq 0$ for conserved charge part.

Non-equilibrium diagram technique. Matrix presentation

For non-equilibrium state $|n>$:

$$iG^{--}(x,y) =< n|\widehat{T}\widehat{\phi}(x)\widehat{\phi}^{\dagger}(y)|n >=< n|\widehat{S}^{-1}\widehat{T}\widehat{\phi}_I(x)\widehat{\phi}_I^{\dagger}(y)|\widehat{S}|n >$$

$$\neq < 0|\widehat{T}\widehat{\phi}_I(x)\widehat{\phi}_I^{\dagger}(y)|\widehat{S}|0 > /\widehat{S}$$

since non-equilibrium state does not transform to the same state (due to possible decays) ➡ 4 Green functions:

$$iG^{--}(x,y) =< n|\widehat{T}\widehat{\phi}(x)\widehat{\phi}^{\dagger}(y)|n >, \quad iG^{++}(x,y) =< n|\widehat{T^{\dagger}}\widehat{\phi}(x)\widehat{\phi}^{\dagger}(y)|n >,$$

and **Wigner densities** (no time ordering operations):

$$iG^{-+} = \mp < n|\widehat{\phi}^{\dagger}(y)\widehat{\phi}(x)|n >, \quad iG^{+-} =< n|\widehat{\phi}(x)\widehat{\phi}^{\dagger}(y)|n > .$$

Green functions are not independent: ➡ one relation:

$$G^{--} + G^{++} = G^{-+} + G^{+-}.$$

Equations of motion

Assume suppression of initial correlations (for $t \gg t_{cor}$) ➡ Wick theorem.
➡ averaging of operator eqs of motion produces standard (but matrix) Dyson eqs.

$G(x,y) = G^0(x,y) + \int G^0(x,z)\Pi(z,z')G(z',y)$ with

$$iF(x,y) = \begin{pmatrix} iF^{--}(x,y) & iF^{-+}(x,y) \\ iF^{+-}(x,y) & iF^{++}(x,y) \end{pmatrix}, \quad F = \{G, \Pi\}$$

in covariant metric: $F_i^j = \sigma_{ik}F^{kj}, \sigma_{ik} = (\sigma_3)_{ik}, \sigma_i^k = \delta_{ik}, i,k = \{-,+\}$. [We use different notations compared with LP (different signs in G^{ij} and Π^{ij}) and KB text books $(>,<)$.]

(In presence of mean fields we also obtain eq. of motion for the condensate fields.)
Eq. for retarded Green func. $G^R = G^{--} - G^{-+}$ decouples and defines excitation spectrum

$$G^R_{12} = G^{0,R}_{12} + G^{0,R}_{13}\Pi^R_{34}G^R_{42}$$

Thermal equilibrium

In equilibrium only two Green functions are independent:

$$iF(p) = \begin{pmatrix} F^R \pm i n_\epsilon \mathcal{A} & \pm i n_\epsilon \mathcal{A} \\ -i(1 \mp n_\epsilon)\mathcal{A} & -F^A \mp i n_\epsilon \mathcal{A} \end{pmatrix}, \quad \mathcal{A} = A, \text{ for } G \text{ and } \Gamma \text{ for } \Pi.$$

Particle occupations are $n_\epsilon = \frac{1}{e^{(\epsilon/T)} \pm 1}$.

$G^{-+} \propto n_\epsilon$. Spectral function: $A(p) = -2\mathrm{Im}G^R = \frac{\Gamma}{M^2 + \Gamma^2/4}$,

$M = -Q(p) + \mathrm{Re}\Pi$, $Q = \epsilon - \frac{\boldsymbol{p}^2}{2m}$ for non-rel. part.; $Q = -m^2 + p^2$ for rel. bosons.

- Classical limit: $n_\epsilon \ll 1$:

$$iG(p) \rightarrow \begin{pmatrix} G^R & \pm i n_\epsilon A \\ -iA & -G^A \end{pmatrix}$$

➡ only G^{-+} depends on occupations! ➡ Specific role of G^{-+}.

- Quasiparticle limit $\Gamma \rightarrow 0$: $A \rightarrow 2\pi\delta(M)$ that fixes in-medium mass-shell.
- Gas limit: on-mass-shell part.: $\mathrm{Re}\Pi^R \rightarrow 0$, $f(\boldsymbol{p}) = \int iG^{-+} \frac{d\epsilon}{2\pi}$ Boltzmann distrib.

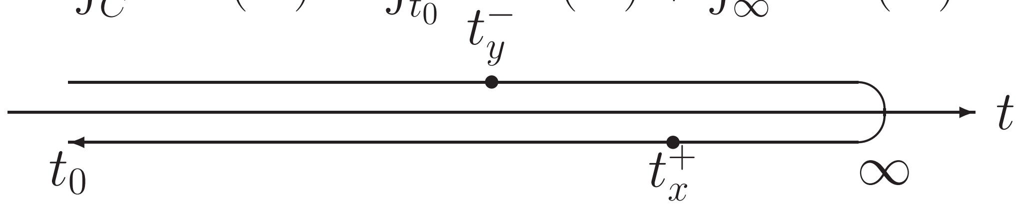
➡ G^{-+} have a meaning of generalized distributions of virtual particles.

No relation between ϵ and $\boldsymbol{p}$(!). We use formulation of all Stat. Phys. quantities in 4-momentum space., not in 3-momentum space, as usually done.

Schhwinger-Keldysh contour formalism

One may introduce instead of matrix notation a closed real-time contour with two external points x, y on the contour. For any two-point funct. $F^{ij}(x,y) = F(x^i, y^j)$,

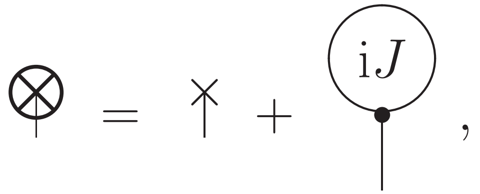
$$\int_C dx^C(\ldots) = \int_{t_0}^{\infty} dx^-(\ldots) + \int_{\infty}^{t_0} dx^+(\ldots):$$

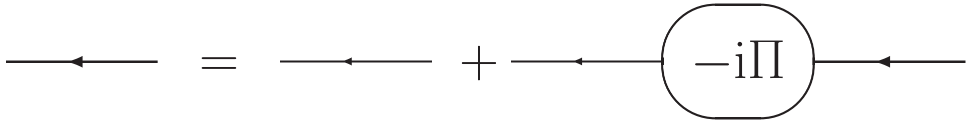


$$iG_C(x,y) =< T_C \widehat{\varphi}(x)\widehat{\varphi}^\dagger(y) >=< T_C \widehat{\phi}(x)\widehat{\phi}^\dagger(y) > -\phi(x)\phi^*(y),$$

$$\delta_C(x,y) = \sigma^{ij}\delta^4(x-y),\ \phi \text{ are mean fields.}$$

Averaging of operator eqs yields eqs for mean fields and Dyson eqs on real contour:





$S_x G_C(x,y) = \delta_C(x,y) + \int_C dz \Pi(x,z) G(z,y)$,: with the one- and two-point functions i$J(x)$ and $-$i$\Pi(x,y)$, as driving terms.

Generating functional W

cf. J.M.Luttinger and J.C.Ward, Phys. Rev. **118** (1960) 1417; J. Cornwell, R. Jackiw, E. Tomboulis, Phys. Rev. **D10** (1974) 2428 for ground state field theory.

We do extension for contour, mean fields, scaled vertices (to derive energy-momentum):

$$\widehat{\mathcal{L}}^{\rm int}_{\lambda} = \lambda(x)\widehat{\mathcal{L}}^{\rm int}\left\{\widehat{\phi}^{\dagger}(x), \widehat{\phi}(x)\right\},$$

$\lambda(x)$ is a space-time dependent interaction scale.

The generating functional then reads

$$W\left\{\eta, K, \lambda\right\} = -\mathrm{i}\ln\left\langle \exp\left[-\mathrm{i}\int dx\,\widehat{H}^{0}_{I}\right]\mathcal{T}_{\mathcal{C}}\exp\left\{\mathrm{i}\int_{\mathcal{C}} dx\left(\lambda(x)\widehat{\mathcal{L}}^{\rm int}_{\rm I}\right.\right.\right.$$
$$\left.\left.\left.+\left(\frac{1}{2}\right)_{\rm neut.}\left[\eta(x)\widehat{\phi}^{\dagger}_{\rm I}(x) + \eta^{*}(x)\widehat{\phi}_{\rm I}(x) + \mathrm{i}\int_{\mathcal{C}} dy\,\widehat{\phi}_{\rm I}(x)K(x,y)\widehat{\phi}^{\dagger}_{\rm I}(y)\right]\right)\right\}\right\rangle,$$

with the free Hamiltonian $\widehat{H}^{0}_{\rm I}$, and one- and two-point sources η and K.

In brief notation $\left(\frac{1}{2}\right)_{\rm neut.} \equiv \sum_{a}^{*}$, and part. index "$a$" is suppressed.

Double-Legendre transformed functional Γ

The double Legendre transformation of W yields:

$$\Gamma\{\phi,\phi^*,G,\lambda\} \;=\; W\{\eta,K,\lambda\} - \left(\frac{1}{2}\right)_{\text{neut.}} \left[\int_{\mathcal{C}} dx\,[\eta^*(x)\phi(x)+\eta(x)\phi^*(x)] \right.$$
$$\left. + \int_{\mathcal{C}} dxdy\,[\phi(y)\phi^*(x)+G(y,x)]\,K(y,x)\right].$$

Here, one- and two-point sources η and K have to be expressed through ϕ and G.

Γ provides eqs of motion for classical fields as well as Dyson's eqs

$$\delta\Gamma/\delta\phi_a = 0,\;\; \delta\Gamma/\delta\phi_a^* = 0,\;\; \delta\Gamma/\delta G_a = 0.$$

At vanishing external sources $\eta = K = 0$

$$\delta\Gamma/\delta\lambda = -\epsilon_\lambda^{int}(x) = -< L^{int} > /\lambda, \quad \Gamma = W.$$

Generating functional Φ (for J and Π)

Generalization of equilibrium expressions introduced by Luttinger, Ward, Baym (1961)

We represent our real-time Γ-functional in the form (thus defining Φ functional!)

$$\Gamma\{\phi,\phi^*,G,\lambda\} = \Gamma^0 + \int_{\mathcal{C}} dx \mathcal{L}^0\{\phi,\partial_\mu\phi\}$$
$$+\mathrm{i}\left(\frac{1}{2}\right)_{\text{neut.}} \left[\ln\left(1-\odot G^0\odot\Pi\right)+\odot G\odot\Pi\right]+\Phi\{\phi,\phi^*,G,\lambda\},$$

$$\odot G\odot\Pi = \int_C dxdy \Pi(x,y)G(y,x),$$

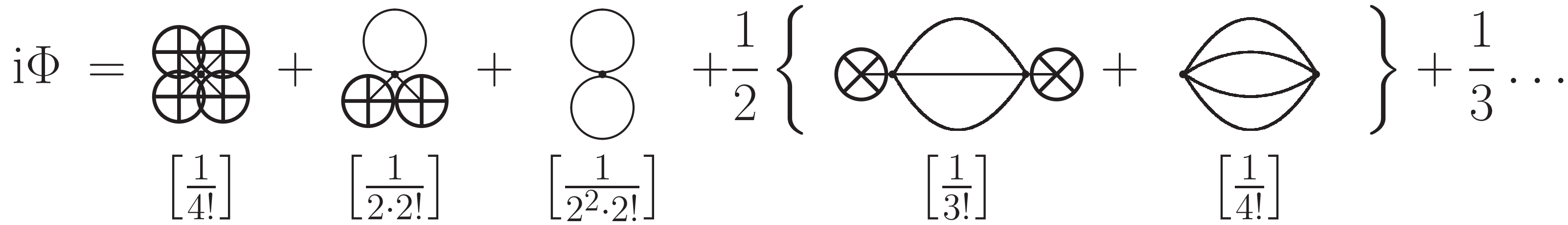
$$\mathrm{i}J(x) = \frac{\delta\mathrm{i}\Phi}{\delta\phi^*(x)}, \quad -\mathrm{i}\Pi(x,y) = \frac{\delta\mathrm{i}\Phi}{\delta\mathrm{i}G(y,x)}\times\begin{cases}2 & \text{for real fields}\\ 1 & \text{for complex fields}\end{cases}$$
$$-\mathcal{E}^{\mathrm{int}}(x) = \frac{\delta\mathrm{i}\Phi}{\delta\mathrm{i}\lambda(x)},$$

G,ϕ,ϕ^* are solutions of eqs of motion, λ is put 1 after variation.

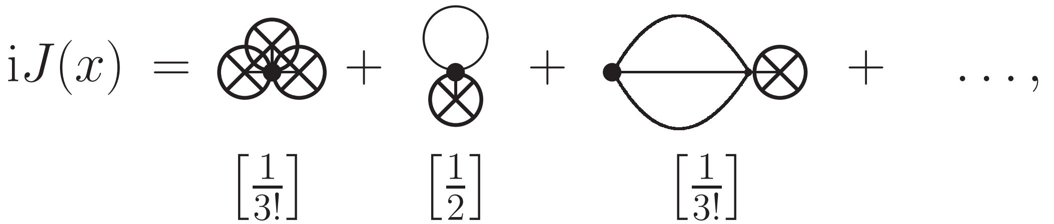
Approximation schemes can be defined through a particular approximation to Φ.

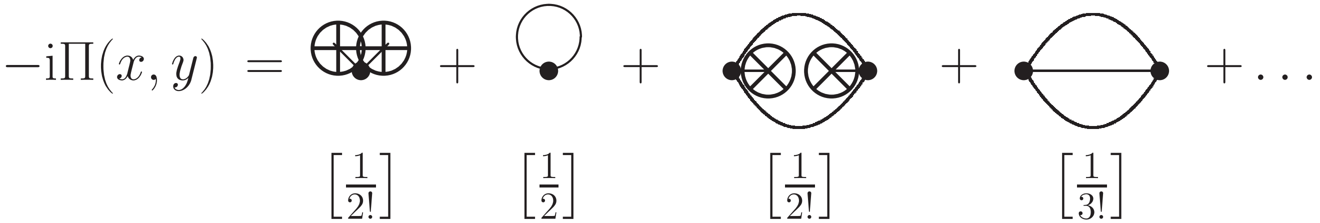
Diagrammatic representation of Φ

Example: $L^{int} = \lambda \widehat{\phi}^4/4!$



The $1/n_\lambda$ factors are explicitly given, the combinatorial factors are in square brackets.
Functional derivatives of Φ with respect to ϕ (pins) and propagators **(full lines)**, determine the source $J(x)$ of the classical field and the self-energy $\Pi(x, y)$, respectively,





Φ derivable approximation scheme

Advantage: any order Φ derivable scheme is conserving:

$$\mathrm{i}J_a^{\text{(appr.)}}(x) = \frac{\delta \mathrm{i}\Phi^{\text{(appr.)}}}{\delta\left(\phi_a^{\text{(appr.)}*}(x)\right)},$$

$$-\mathrm{i}\Pi_a^{\text{(appr.)}}(x,y) = (\mp)_a \frac{\delta \mathrm{i}\Phi^{\text{(appr.)}}}{\delta \mathrm{i}G_a^{\text{(appr.)}}(y,x)} \times \begin{cases} 2 & \text{for neutral fields} \\ 1 & \text{for charged fields} \end{cases}$$

From "approximate" eqs of motion for classical fields and Dyson eqs one recovers mean fields $\phi_a^{\text{(appr.)}}$ and (full but approx.) $G_a^{\text{(appr.)}}$.

Invariance of Φ and identities. $U(1)$ invariance

The $U(1)$ invariance of Φ ($\delta\Phi = 0$) under variation of classical fields and propagators:

$$\phi(x) \Rightarrow e^{-\mathrm{i}e\Lambda(x)}\phi(x), \quad \phi^*(x) \Rightarrow e^{\mathrm{i}e\Lambda(x)}\phi^*(x), \quad G(x,y) \Rightarrow e^{-\mathrm{i}e\Lambda(x)}G(x,y)e^{\mathrm{i}e\Lambda(y)}$$

to linear order in the phase Λ produces the identity

$$e\left\{[J^*(x)\phi(x) - J(x)\phi^*(x)] + (\mp)_a \mathrm{i}\int_{\mathcal{C}} dy\,[\Pi(x,y)G(y,x) - G(x,y)\Pi(y,x)]\right\} = 0,$$

$(\pm)_a = +$ for fermions and $(\pm)_a = -$ for bosons,
$\partial_\mu < \widehat{j}_\mu^{Noether} >= 0$ with the Noether current

$$j^\mu(x) =: \left\langle \widehat{j}^{\,\mu}_{\text{(Noether)}} \right\rangle = e\left[\left((\partial_x^\mu)^* + \partial_y^\mu\right)\left(\phi^*(x)\phi(y) + \mathrm{i}G^{\text{sym}}(y,x)\right)\right]_{x=y}$$

In order to keep $\Theta^{\mu\nu}$ and the Noether current j^μ charge symmetric we introduce

$$F^{\text{sym}}(x,y) = \frac{1}{2}\left(F(x^-,y^+) + F(x^+,y^-)\right),$$

where $F(x,y)$ is any two-point function on the real-time contour.

Invariance of Φ and identities. Space-time invariance

Space-time variation of Φ:

$$
\begin{aligned}
\phi(x) &\Rightarrow \phi(x+\xi(x)) \\
G(x,y) &\Rightarrow G(x+\xi(x), y+\xi(y)), \\
\lambda(x) = 1 &\Rightarrow \lambda(x) = \det\left(\delta_\mu^\nu + \frac{\partial \xi_\mu}{\partial x_\nu}\right), \quad \text{i.e. } \delta\lambda(x) = \frac{\partial \xi_\mu}{\partial x_\mu}
\end{aligned}
$$

also leaves Φ invariant. This way, one deduces the following identity

$$
\begin{aligned}
\frac{\partial}{\partial x_\mu}\mathcal{E}^{\text{int}}(x) &+ \left(\frac{1}{2}\right)_{\text{neut.}} \left[J^*(x)\frac{\partial \phi(x)}{\partial x_\mu} + J(x)\frac{\partial \phi^*(x)}{\partial x_\mu}\right] \\
&+ (\mp)_a \mathrm{i} \left(\frac{1}{2}\right)_{\text{neut.}} \int_{\mathcal{C}} dy \left[\Pi(x,y)\frac{\partial G(y,x)}{\partial x_\mu} + \frac{\partial G(x,y)}{\partial x_\mu}\Pi(y,x)\right] = 0.
\end{aligned}
$$

Invariance of Φ and identities. Space-time invariance

and finds the conserving ($\partial_\mu \Theta^{\mu\nu}(x) = 0$) Noether energy-momentum tensor

$$\Theta^{\mu\nu}(x) = \frac{1}{2}\left(\frac{1}{2}\right)_{\text{neut.}} \left[\left((\partial_x^\nu)^* + \partial_y^\nu \right) \left((\partial_x^\mu)^* + \partial_y^\mu \right) \left(\phi^*(x)\phi(y) + \mathrm{i}G^{\text{sym}}(y,x) \right) \right]_{x=y}$$
$$+ g^{\mu\nu} \left(\mathcal{E}^{\text{int}}(x) - \mathcal{E}^{\text{pot}}(x) \right)$$

where the interaction energy is related to Φ as $\mathcal{E}^{\text{int}}(x) = \delta\Phi/\delta\lambda(x)$; and the potential energy density as:

$$\mathcal{E}^{\text{pot}}(x) = \frac{1}{2}\left(\frac{1}{2}\right)_{\text{neut.}} \Bigg\{ - \left[J^*(x)\phi(x) + J(x)\phi^*(x) \right]$$
$$+ \mathrm{i} \int_{\mathcal{C}} dz \left[\Pi(x,z)G(z,x) + G(x,z)\Pi(z,x) \right] \Bigg\}.$$

All is expressed in terms of Φ derivable mean fields, Green functions and self-energies.

Thermal equilibrium

In the thermal equilibrium the thermodynamic potential is explicitly known,

$$\Omega = -T \ln Z, \quad \text{where} \quad \widehat{\rho}^{\rm eq} = \frac{\exp\left(-\beta \widehat{H}\{\mu\}\right)}{Z}, \quad \text{with} \quad {\rm Tr}\, \widehat{\rho}^{\rm eq} = 1,$$

where $\beta = 1/T$ is the inverse temperature, $\widehat{\rho}^{\rm eq}$ is now the **equilibrium** density operator and Z is the partition function.

In W-functional we now explicitly write ${\rm Tr}\, \widehat{\rho}^{\rm eq}...$ instead of $\langle ... \rangle$. Thus, taking into account that $\Gamma_{\rm eq} = W$ at vanishing external sources, we arrive at the Γ functional in equilibrium $\Gamma = \Gamma_{\rm n.eq} + \Gamma_{\rm eq}$ with the integration contour $\mathcal{C}_{\rm eq}$ now being the sum of the real-time Schwinger-Keldysh contour and the imaginary-time Matsubara contour, i.e. it starts from **an initial time** t_0**,** goes to infinity, then back to this initial time and after that, to $t_0 - {\rm i}\beta$:

$$\Gamma^{\rm eq}\{\phi, \phi^*, G, \lambda, \mu\} = -{\rm i} \ln \left(\frac{1}{Z} {\rm Tr} \exp\left[-{\rm i} \int_{\mathcal{C}_{\rm eq}} dt\, \widehat{H}^0_{\rm I}\{\mu\} \right] \mathcal{T}_{\mathcal{C}} \exp\left[{\rm i} \int_{\mathcal{C}_{\rm eq}} dx \lambda \widehat{\mathcal{L}}^{\rm int}_{\rm I}\{\mu\} \right] \right).$$

Thermodynamic consistency

Using eqs of motion $\Gamma^{eq} = 0$ for the physical values of ϕ, ϕ^*, G, and x-independent λ, we obtain Z and

$\Omega\{\phi, \phi^*, G, \lambda, \mu\} = -T \ln Z$, that provides a thermodynamic potential expressed in terms of the real-time quantities

$$\Omega\{\phi, \phi^*, G, \lambda, \mu\} = -\int d^3x \mathcal{L}^0\{\phi, \partial_\mu\phi\} + \left(\frac{1}{2}\right)_{\text{neut.}} \int d^3x \frac{d^4p}{(2\pi)^4} n(\varepsilon)$$
$$\times \left(-2\mathrm{Im}\ \ln\left[-G^R(\varepsilon + i0, \boldsymbol{p})\right] - \mathrm{Re}G^R\Gamma - A\mathrm{Re}\Pi^R\right) + \Phi_T.$$

Thus we obtained a non-equilibrium $\Gamma = -\int_C dt \Omega^{n.eq.}$ with correct equilibrium limit.
$\Omega^{\text{appr.}}$ is consistent with $E^{\text{appr.}}$, S^{appr}, $F^{\text{appr.}}$, etc. ➡
Thermodynamic consistency.

Intermediate conclusions

- Scheme can be used in description of strong interacting off-mass-shell particle (resonance) matter, including possible phase transitions.
- We constructed generating functional Φ in terms of closed skeleton diagrams on real-time contour.

  $\Phi[\phi, G]$ plays similar role as $L^{int}[\widehat{\phi}, \partial_\mu \widehat{\phi}]$ for initial field theory $\Rightarrow$ effective theory.
- Φ derivable approximations are **exactly** conserving: j_μ, $\theta^{\mu\nu}$ coincide with Noether quantities. We derived consistency identities being fulfilled in Φ derivable schemes.

- Φ derivable approximations are thermodynamically consistent. At equilibrium Φ_T is reduced to that previously constructed by Luttinger, Ward, Baym.

Transport scheme. Wigner transformation

Separate slow space-time macro-motion $X = \frac{1}{2}(x_1 + x_2)$ and rapid relative motion $\xi = x_1 - x_2$ related to micro- processes, e.g. for $t_{cor} \ll t_{micro} \sim 1/\epsilon_F \ll 1/\Gamma$. For any two-point function

$$F^{ij}(X;p) = \int d\xi e^{\mathrm{i}p\xi} F^{ij}\left(X + \xi/2, X - \xi/2\right), \qquad i,j \in \{-+\}$$

Gradient approximation (for $\Delta p_\mu \Delta X^\mu \gg \bar{h}$) converts

$$\int d\xi e^{\mathrm{i}p\xi}\left(\int dz f(x,z)\varphi(z,y)\right) = \left(\exp\left[\frac{\mathrm{i}\bar{h}}{2}\left(\partial_p \partial_{X'} - \partial_X \partial_{p'}\right)\right] f(X,p)\varphi(X',p')\right)_{p'=p, X'=X}$$

$$\simeq f(X,p)\varphi(X,p) + \frac{\mathrm{i}\bar{h}}{2}\left\{f(X,p), \varphi(X,p)\right\},$$

where the first order terms are given by Poisson brackets

$$\{f(X,p), \varphi(X,p)\} = \frac{\partial f}{\partial p^\mu}\frac{\partial \varphi}{\partial X_\mu} - \frac{\partial f}{\partial X^\mu}\frac{\partial \varphi}{\partial p_\mu}$$

Kadanoff-Baym eqs.

The Wigner transformation of Dyson eqs yields so-called Kadanoff-Baym eqs:

$$
\begin{aligned}
\mathrm{i}v_\mu \partial_X^\mu G^{ij}(X,p) &= \int d\xi e^{\mathrm{i}p\xi} \int_{\mathcal{C}} dz \left(\Pi(x^i,z)G(z,y^j) - G(x^i,z)\Pi(z,y^j)\right), \\
\widehat{Q}_X G^{ij}(X,p) &= \sigma^{ij} + \frac{1}{2}\int d\xi e^{\mathrm{i}p\xi} \int_{\mathcal{C}} dz \left(\Pi(x^i,z)G(z,y^j) + G(x^i,z)\Pi(z,y^j)\right)
\end{aligned}
$$

$v^\mu = (1, \boldsymbol{p}/m)$, and $\widehat{Q}_X = p_0 - \boldsymbol{p}^2/2m - \Delta/8m$ for non-rel. part. and $v^\mu = 2p^\mu$ and $\widehat{Q}_X = -m^2 + p^2 - \partial_x^2$ for rel. bosons.

We have two extra Dyson eqs. (8 contour-integral-differential eqs for real quantities).
A simplification: Particular combinations of these equations lead to the retarded and advanced eqs which decouple and involve only integrations over physical times rather than contour times. ➡ Still very complicated!

Physical notations for kinetic quantities

It is helpful to avoid the imaginary factors inherent in the standard Green function formulation and introduce quantities which are real and, in the quasi-homogeneous limit, positive, and therefore have a straightforward physical interpretation, much like for the Boltzmann equation. We define

$$\begin{aligned} F(X,p) &= A(X,p)f(X,p) = (\mp)\mathrm{i}G^{-+}(X,p), \\ \widetilde{F}(X,p) &= A(X,p)[1 \mp f(X,p)] = \mathrm{i}G^{+-}(X,p), \end{aligned}$$

for the generalized Wigner functions F and $\widetilde{F}$ and the corresponding *four*-phase-space distribution functions $f(X,p)$ and Fermi/Bose factors $[1 \mp f(X,p)]$. Here

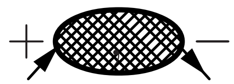
$$A(X,p) \equiv -2\mathrm{Im}G^R(X,p) = \widetilde{F} \pm F = \mathrm{i}\left(G^{+-} - G^{-+}\right)$$

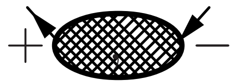
is the spectral function in Wigner representation.

Problem: According to relations between Green functions G^{ij}, only two independent real functions of all the G^{ij} are required for a complete kin. description. But we have 8 eqs for real quantities (!). ➡ 6 eqs are not independent.

Physical notations for kinetic quantities

Likewise the reduced gain and loss rates of the collision integral are defined as

$$\Gamma_{in}(X,p) = \Gamma(X,p)\gamma(X,p) = (\mp)\mathrm{i}\Pi^{-+}(X,p) =$$ 

$$\Gamma_{out}(X,p) = \Gamma(X,p)[1 \mp \gamma(X,p)] = \mathrm{i}\Pi^{+-}(X,p)$$ 

with the damping width

$$\Gamma(X,p) \;\equiv\; -2\mathrm{Im}\Pi^{R}(X,p) = \Gamma_{out}(X,p) \pm \Gamma_{in}(X,p),$$

The opposite combination

$$I(X,p) = [2\gamma \mp 1]\Gamma = \Gamma_{in}(X,p) \mp \Gamma_{out}(X,p),$$

is related to fluctuations.

Generalized kinetic eq. in physical notation

Within the first-order gradient approximation, the first KB eq. yields now two eqs

$$DF(X,p) - \left\{\Gamma_{in}, \mathrm{Re}G^R\right\} = C(X,p),$$
$$D\widetilde{F}(X,p) - \left\{\Gamma_{out}, \mathrm{Re}G^R\right\} = \mp C(X,p),$$

which we denote as the generalized transport (kin.) eqs. Here the diff. drift operator

$$D = \left(v_\mu - \frac{\partial \mathrm{Re}\Pi^R}{\partial p^\mu}\right)\partial_X^\mu + \frac{\partial \mathrm{Re}\Pi^R}{\partial X^\mu}\frac{\partial}{\partial p_\mu}$$ describes drag flow in a mean field

The Poisson bracketed term describes back flow due to fluctuations.

Cf. a toy-ship moving in a bath.

Case $C = 0$: Vlasov collision-less dynamics. Also $C = 0$ is fulfilled for thermal equilibrium when $\partial_X F = 0$. The collision term:

$$C(X,p) = \Gamma_{in}(X,p)\widetilde{F}(X,p) - \Gamma_{out}(X,p)F(X,p) = A\Gamma[\gamma - f].$$

It can be expressed in terms of Φ, thus preserving all symmetries of Φ:

$$C(X,p) = \frac{\delta \mathrm{i}\Phi}{\delta \widetilde{F}(X,p)}\widetilde{F}(X,p) - \frac{\delta \mathrm{i}\Phi}{\delta F(X,p)}F(X,p).$$

$F(X,p)$ is 4-phase-space probability-a generalized virtual part. distrib.

Clearly we deal with a generalized kinetic eq.!

Other eqs

The second KB eq. provides us with two alternative equations for F and $\widetilde{F}$:

$$\begin{aligned} MF - \mathrm{Re}G^R\Gamma_{in} &= \frac{1}{4}\left(\{\Gamma, F\} - \{\Gamma_{in}, A\}\right), \\ M\widetilde{F} - \mathrm{Re}G^R\Gamma_{out} &= \frac{1}{4}\left(\left\{\Gamma, \widetilde{F}\right\} - \{\Gamma_{out}, A\}\right) \end{aligned}$$

with the "mass" function $M(X,p) = p_0 - \frac{1}{2m}\boldsymbol{p}^2 - \mathrm{Re}\Pi^R(X,p)$ (for non-rel. part.), and $M(X,p) = p^2 - \mathrm{Re}\Pi^R(X,p)$ (for rel. bosons), which relates to the drift operator via $Df = \{M, f\}$ for any four-phase-space function f. These eqs can be called the generalized mass-shell equations, since in the quasiparticle (qp) limit they provide the mass-condition $M = 0$. Yet they have no clear interpretation beyond qp. approx.

Combinations of the two sets of eqs provide us with the algebraic (!) retarded eqs

$$G^R = \frac{1}{M(X,p) + \mathrm{i}\Gamma(X,p)/2} \Rightarrow \left\{ \begin{aligned} A(X,p) &= \frac{\Gamma(X,p)}{M^2(X,p) + \Gamma^2(X,p)/4}, \\ \mathrm{Re}G^R(X,p) &= \frac{M(X,p)}{M^2(X,p) + \Gamma^2(X,p)/4}, \end{aligned} \right.$$

satisfying the sum–rule $\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} A(X,p) = 1$, for non-rel. part., , and $\int_{-\infty}^{\infty} p_0 \frac{dp_0}{2\pi} A(X,p) = 1$, for rel. bosons. Thus retarded eqs get clear physical meaning.

Other eqs

• Two extra Dyson eqs for G^{--} and G^{++} in the first order gradient expansion produce the same 6 eqs. (being in a line with one relation between Green functions for complex quantities.) Thus we have two independent quantities, e.g., A and f, and 6 eqs still. Then we have:

• Kramers-Kronig relation between ReG^R and ImG^R. ➡ $A = F(\mathrm{Re}G^R)$.

• Using F (or f) and A as independent variables it is easy to show that only one kin. eq. among two (for F and $\widetilde{F}$) is independent.

• Similarly, only one mass-shell eq. among two is independent, if one selects F (or f) and A as independent variables.

• Thus we retained with 3 eqs for 2 real quantities: one differential kin. eq. for F (or f), algebraic eq. for A, and one mass-shell eq. for F.

Two strategies: (i) ignore mass-shell eq. and deal only with the kin. eq. and with algebraic eq. for A. We call it Kadanoff-Baym choice., or (ii) to prove approximate equivalence of kin. and mass-shell eqs. We call it Botermans-Malfliet choice., see below.

Quasiparticle limit of kin. eq.

Qp approximation: receipt: take $\Gamma \to 0$ in the Green functions and calculate self-energies with them:

$A^{qp} = 2\pi \mathrm{sgn}\Gamma \delta(M) + O(\Gamma)$. Integrating kin. eq. (i.e. eq. for F) in $d\epsilon$ we arrive at the qp. kin. eq. It is a consistent procedure if evolution does not lead a system beyond the region of small Γ (smaller than all typical energy values involved in the problem). E.g., in Fermi liquids for $T \ll \epsilon_F$, $\Gamma \propto T^2/\epsilon_F \ll \epsilon_F$ and cond-s for qp approx. hold.

To be specific consider a system of non-rel. part.

In general case the drift operator D can be expressed as

$$D = \frac{1}{Z}\left(\partial_t + \boldsymbol{v}_g \partial_{\mathbf{x}}\right) + \partial_t \mathrm{Re}\Pi^R \cdot \partial_{p_0} - \partial_{\mathbf{x}} \mathrm{Re}\Pi^R \cdot \partial_{\boldsymbol{p}} \, ,$$

where

$$\boldsymbol{v}_g(X,p) = Z\left(\boldsymbol{v} + \frac{\partial \mathrm{Re}\Pi^R}{\partial \boldsymbol{p}}\right) \quad \text{and} \quad Z = \left(v_0 - \frac{\partial \mathrm{Re}\Pi^R}{\partial p_0}\right)^{-1}$$

in the qp approx. get the meaning of the group 3-velocity and the standard renormalization factor. (Note: In the generalized kinetics ε and $\boldsymbol{p}$ are independent.)

Within our approach to derive qp limit we do not use any ansatze reductions(!), as KB ansatz, generalized KB ansatz etc., which are usually used to that aim.

Quasiparticle limit

Integrating the generalized kin. eq. in $d\epsilon$ we obtain the standard Landau qp kin. eq.:

$$\partial_t f^{\rm qp}(X,\boldsymbol{p}) + \frac{\partial \varepsilon(X,\boldsymbol{p})}{\partial \boldsymbol{p}} \partial_{\mathbf{X}} f^{\rm qp}(X,\boldsymbol{p}) - \partial_{\mathbf{X}} \varepsilon(X,\boldsymbol{p}) \, \frac{\partial f^{\rm qp}(X,\boldsymbol{p})}{\partial \boldsymbol{p}} = C^{\rm qp}(X,\boldsymbol{p}),$$

$C^{\rm qp}(X,\boldsymbol{p}) \equiv C(F = A^{qp} f^{qp}, \ldots)$. The qp dispersion relation for the energies $\varepsilon(X,\boldsymbol{p})$ follows from the retarded eq.:

$$\varepsilon(X,\boldsymbol{p}) = \frac{1}{2m}\boldsymbol{p}^2 + \mathrm{Re}\, \Pi^R\left(X, \varepsilon(X,\boldsymbol{p}), \boldsymbol{p}\right).$$

This dispersion eq. may have several solutions (branches) which contribute separately.

The fluctuation term being integrated over ϵ is of the order Γ times gradients and therefore is omitted in this lowest order approximation.

Note Boltzmann eq. is obtained in perturb. limit: $\varepsilon(X,\boldsymbol{p}) \simeq \frac{1}{2m}\boldsymbol{p}^2$, with simplest $\propto f f(1-f)(1-f)$ collision term.

Positive definiteness of kin. quantities

For a semi-classical interpretation one likes to have the Wigner distributions $F(X,p)$ and $\widetilde{F}(X,p)$ to be positive, as in Boltzmann case.

Using the operator def. for the Green functions and integrating it:

$$\int dX F(X,p) = \left\langle \left(\int dy e^{ipy} \widehat{\varphi}^\dagger(y) \right) \left(\int dx e^{-ipx} \widehat{\varphi}(x) \right) \right\rangle \geq 0,$$

and similarly for self-energies Π expressed through the current–current correlator. Thus, we get the following set of constraints

$$\int dX \widetilde{F}(X,p) \geq 0, \quad \int dX F(X,p) \geq 0, \quad \int dX \Gamma_{out}(X,p) \geq 0, \quad \int dX \Gamma_{in}(X,p) \geq 0.$$

Similar relations are obtained for the integration over four-momentum space. As a result, in stationary and spatially homogeneous systems, in part. in equilibrium systems:

$$F(p) \geq 0, \quad \widetilde{F}(p) \geq 0; \quad \Gamma_{in}(p) \geq 0, \quad \Gamma_{out}(p) \geq 0.$$

If grad-s are small positivity contin. to hold when system evolves along characteristics.

Local (Markovian) collision term and memory

The collision term is expressed in terms of self-energies and Green functions. Once self-energy diagrams and diagrams of Φ consist of more than 2 vertices they depend on many independent time points $t_1, t_2, t_3, \ldots$ demonstrating memory effects. Thus within the gradient approx. they should be still expanded: $C = C^{loc} + C^{mem}$, C^{loc} is the (Markovian) collision term, evaluated with all Green functions in the Wigner representation taken at the same space-time point X. Using symmetries of Φ and decomposing it in $F\widetilde{F}$ products we present $C_a^{\rm loc}$ as

$$C_a^{\rm loc}(X,p) = \frac{1}{2}\sum_{m,m'}\int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_m}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\cdots\frac{d^4p'}{(2\pi)^4}$$

$$\times R_{m,m'}(X;p_1,\cdots,p_m;p'_1,\cdots,p'_{m'})\left\{\widetilde{F}_1\cdots\widetilde{F}_m F'_1\cdots F'_{m'} - F_1\cdots F_m\widetilde{F}'_1\cdots\widetilde{F}'_{m'}\right\}$$

$$\times\left[\sum_{i=1}^{m}\delta_{aa_i}(2\pi)^4\delta^4(p_i-p) - \sum_{i=1}^{m'}\delta_{aa_i}(2\pi)^4\delta^4(p'_i-p)\right](2\pi)^4\delta^4\left(\sum_{i=1}^{m}p_i - \sum_{i=1}^{m'}p'_i\right).$$

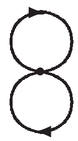
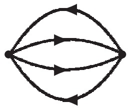
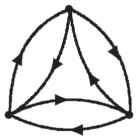
$R_{m,m'}$ have sense of partial process rates(!) This expression visualizes the detailed balance property, that the same multi-particle rate determines both the forward and the backward processes. For cases $R_{m,m'} = R_{m',m}$ we further prove H theorem.

Memory effects in collision term

To be specific consider a system of fermions interacting via a two-body potential $V = V_0\delta(x-y)$, and, for the sake of simplicity, disregard its spin structure, by relating spin and anti-symmetrization effects to a degeneracy factor d,

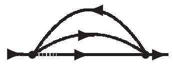
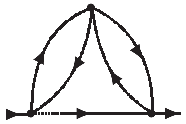
for equilibrium case cf. G.M. Carneiro, C. Pethick: Phys. Rev. **11** (1975) 1106

$$\mathrm{i}\Phi = \mathrm{i}\left(\Phi^{(1)} + \Phi^{(2)} + \Phi^{(3)} + \ldots\right)$$

$= \frac{1}{2}$  $+ \frac{1}{4}$  $+ \frac{1}{6}$  $+ \ldots,$

The corresponding self-energy becomes

$$-\mathrm{i}\Pi = -\mathrm{i}\left(\Pi^{(1)} + \Pi^{(2)} + \Pi^{(3)}\right) =$$

$=$  $+$  $+$  $+ \ldots$

Already 3-vertex term has gradient structure memory effect in kin. description!

Memory effects in collision term up to 3 vertices

The local part C^{loc} can easily be derived in the form

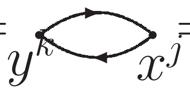
$$C^{(2)} + C^{(3)}_{\text{loc}} = d^2 \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p_3}{(2\pi)^4} \Big(\Big|$$  $$+$$  $$\Big|^2 - \Big|$$  $$\Big|^2 \Big)$$

$$\times (2\pi)^4 \delta^4 (p + p_1 - p_2 - p_3) \left(F_2 F_3 \widetilde{F} \widetilde{F}_1 - \widetilde{F}_2 \widetilde{F}_3 F F_1 \right), \qquad (1)$$

where all vertices in the off-shell rates are of the same sign, say "-". Positivity the rate is not obvious. A subtle question (!): A self-consistency of approx. is very important. The first-order gradient correction to the collision term (memory):

$$C^{(3)}_{\text{mem}}(X,p) = \left[\left(\Pi^{(3)}_{+-} \right)_{\text{mem}} (X,p) G^{-+}(X,p) - G^{+-}(X,p) \left(\Pi^{(3)}_{-+} \right)_{\text{mem}} (X,p) \right]$$
$$= \frac{\mathrm{i}}{2} \int \frac{d^4 p'}{(2\pi)^4} \frac{1}{d} \left[\widetilde{L}^{+-}(X; p'+p, p) - \widetilde{L}^{-+}(X; p'+p, p) \right] \left\{ L^{+-}, L^{-+} \right\}_{p',X}. \qquad (2)$$

with the Wigner transformed the loop function (Lindhard function)

$$L^{jk}(x,y) =$$  $$= d \mathrm{i} V_0 \mathrm{i} G^{jk}(x,y) \mathrm{i} G^{kj}(y,x). \qquad (3)$$

Conservations of charge and energy–momentum

Generalized kin. eq. weighted either with the charge e or with 4-momentum p^ν, integrated over momentum and summed over internal degrees of freedom like spin (Tr) gives rise to the charge or energy–momentum conservation laws, respectively. E.g. for the current we thus derive:

$$j_\mu(X) = e\mathrm{Tr}\int \frac{d^4p}{(2\pi)^4}\left(v_\mu - \frac{\partial \mathrm{Re}\Pi^R}{\partial p^\mu}\right) F(X,p) - \mathrm{Re}G^R \frac{\partial(\Gamma_{in} \mp \Gamma_{out})}{\partial p^\mu} + j_\mu^{mem},$$

First term contains Noether piece and the Markovian drag current. Second term is Markovian back current; j_μ^{mem} ($j_{\mu(3)}^{mem} = 0$, $j_{\mu(4)}^{mem} \neq 0$) is the memory current. Drag, back and memory currents exactly cancel each other if consistency conditions are fulfilled, i.e. in a self-consistent Φ derivable scheme.

Consistency conditions

We have demonstrated that before grad. expansion conservation laws hold exactly only if all the self-energies are Φ-derivable. After Wigner transformation and first order gradient expansion consistency relations read

$$\mathrm{iTr}\left(\frac{1}{2}\right)_{\mathrm{neut.}} \int \frac{d^4p}{(2\pi)^4} \left[\left\{\mathrm{Re}\Pi^R, F\right\} - \left\{\mathrm{Re}G^R, \Gamma_{in}\right\} + C\right] = 0,$$

for the conserved current and

$$\partial^\nu \left(\mathcal{E}^{\mathrm{int}} - \mathcal{E}^{\mathrm{pot}}\right) = \mathrm{Tr} \int \frac{p^\nu d^4p}{(2\pi)^4} \left[\left\{\mathrm{Re}\Pi^R, F\right\} - \left\{\mathrm{Re}G^R, \Gamma_{out}\right\} + C\right]$$

for the energy-momentum tensor.

Question: Identities holden exactly before grad. expansion. Do they remain exact after first order grad. expansion or do they hold up to second order gradients?

Ref. J. Knoll, Yu.B. Ivanov, D.V.: Ann. Phys. **293** (2001) 126. introduced a specific diagram technique for the treatment of gradients of G and Π and has shown that only a unified treatment of both the Poisson brackets (fluctuations) and memory collision term reveals the symmetry along these terms, such that the conservation laws for the Noether current and energy-momentum are **exactly** conserved!

Conservations of charge and energy–momentum

We derive simple expressions for the Noether 4-current and Noether energy–momentum tensor:

$$
\begin{aligned}
j^{\mu}(X) &= e\mathrm{Tr}\int \frac{d^4p}{(2\pi)^4} v^{\mu} F(X,p), \\
\Theta^{\mu\nu}(X) &= \mathrm{Tr}\left(\frac{1}{2}\right)_{\text{neut.}} \int \frac{d^4p}{(2\pi)^4} v^{\mu} p^{\nu} F(X,p) + g^{\mu\nu}\left(\mathcal{E}^{\text{int}}(X) - \mathcal{E}^{\text{pot}}(X)\right).
\end{aligned}
$$

being exactly conserved (!) (in KB choice).

$$
\mathcal{E}^{\text{int}}(X) = \left\langle -\widehat{\mathcal{L}}^{\text{int}}(X)\right\rangle = \frac{\delta\Phi}{\delta\lambda(x)}\Big|_{\lambda=1} \text{ and } \mathcal{E}^{\text{pot}} = \mathrm{Tr}\left(\frac{1}{2}\right)_{\text{neut.}} \int \frac{d^4p}{(2\pi)^4} \left[\mathrm{Re}\Pi^R F + \mathrm{Re}G^R \Gamma_{in}\right].
$$

What to do with mass-shell eq.? BM choice.

Considering small deviations from equilibrium Botermans and Malfliet (1990) (see Phys.Rep. **198** 115) suggested to replace Γ_{in} in the first-order gradient Poisson bracketed terms to its equilibrium value using that $\Gamma_{in} = \Gamma F/A + O(\partial_X)$. The correction $O(\partial_X)$ leads then to terms second order in gradients. We call it **BM choice.**

Kin. eq. then becomes

$$DF(X,p) - \left\{ \Gamma \frac{F}{A}, \mathrm{Re}\Delta^R \right\} = C(X,p),$$

Then one can show that the kin. and mass-shell eqs are *exactly* identical to each other and can be presented in the form

$$\frac{A^2\Gamma}{2} \left(Df - \frac{M}{\Gamma} \{\Gamma, f\} \right) = C.$$

where a peaked function $\frac{A^2\Gamma}{2} \to \delta(M)$ in the limit $\Gamma \to 0$. Also we recognize particle drag flow via D and back flow in Poisson bracketed term. ➡ **clear physical interpret.**

Conservation of current and energy-momentum in BM choice

The consistency conditions hold approximately in BM choice, up to second order gradients!

The kin. eq. in BM choice exactly conserves the effective current

$$j_\mu^{BM-ef}(X) = e\mathrm{Tr}\int \frac{d^4p}{(2\pi)^4}\left[v_\mu F(X,p) + \mathrm{Re}\Pi^R \frac{\partial F}{\partial p^\mu} - \mathrm{Re}G^R \frac{\partial(\Gamma F/A)}{\partial p^\mu}\right] + j_\mu^{mem},$$

which differs from the Noether current in terms $O(\partial_X)$ provided a Φ derivable approx. is used. (The term $O(\partial_X)$ in current corresponds to $O(\partial_X^2)$ in conserv. law).

In BM choice Noether current and energy-momentum as well as consistency conditions are recovered within first-order gradient approximation, whereas in KB choice they are exact. However mass-shell eq. differs from the kin. eq. in second gradients in KB choice.

Entropy

Intuitively a quantity, which relates macroscopic to microscopic aspects of nature (as time reversibility and irreversibility) and possesses a well defined classical limit expression, is welcome to measure the amount of chaos. Since degree of chaoticity is usually increased with the time evolution of closed many-particle systems, this quantity should also determine the direction of the evolution of macro- systems to equilibrium.

However there is no unique way how to introduce entropy. This is because the entropy is not an observable. Does not exist quantum mechanical operator which eigenvalue is entropy. Therefore, entropy does not show up in fundamental equations of motion, such as Schrödinger and Dyson equations, and can only be introduced using intuitive indirect arguments. Thereby, there were introduced a tremendous variety of entropy-like quantities, which relation to each other is clear only for specially prepared systems, like weakly interacting rare gas, but not for arbitrary systems. We mention here: von Neuman entropy, information Shannon entropy, kinetic Boltzmann entropy, relative entropy, Segal entropy, etc. Important property: each step of inform. loss increases the entropy. We will derive the kinetic entropy from generalized kin.eq. and will show that in equilibrium it coincides with the von Neumann entropy

$$S = -\mathrm{Tr}\, \widehat{\rho}_{eq} \ln \widehat{\rho}_{eq}.$$

Markovian entropy flow and H theorem

We multiply kin. eq. for F by $(\mp)\ln(\widetilde{F}/A)$, kin. eq. for $\widetilde{F}$ by $-\ln(F/A)$, take their sum, integrate it over $d^4p/(2\pi)^4$ and finally sum the result over internal degrees of freedom like spin (Tr). For BM choice we find

$$\partial_\mu s^\mu_{\rm loc}(X) = \mathrm{Tr}\sum_a \int \frac{d^4p}{(2\pi)^4} \ln\frac{\widetilde{F}_a}{F_a} C_a(X,p),$$

Let suppress gradient corrections to the collision term (i.e. $C_{\rm mem}$) (Markovian approx.):

$$\mathrm{Tr}\int \frac{d^4p}{(2\pi)^4}\ln\frac{\widetilde{F}}{F}C_{\rm loc}(X,p) = \mathrm{Tr}\sum_{m,m'}\frac{1}{2}\int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_m}{(2\pi)^4}\frac{d^4p'_1}{(2\pi)^4}\cdots\frac{d^4p'_m}{(2\pi)^4}$$

$$\times\left\{F_1\cdots F_m\widetilde{F}'_1\cdots\widetilde{F}'_{m'} - \widetilde{F}_1\cdots\widetilde{F}_m F'_1\cdots\widetilde{F}'_{m'}\right\}\ln\frac{F_1\cdots F_m\widetilde{F}'_1\cdots\widetilde{F}'_{m'}}{\widetilde{F}_1\cdots\widetilde{F}_m F'_1\cdots F'_{m'}}$$

$$\times R_{m,m'}\,(2\pi)^4\delta^4\left(\sum_{i=1}^{m}p_i - \sum_{i=1}^{m'}p'_i\right) \geq 0.$$

Since $(x-y)\ln(x-y)\geq 0$, in the case when all rates $R_{m,m'}$ are non-negative ➡ H theorem: sufficient but not necessary condition of reaching equilibrium.

Markovian entropy flow

The local (Markovian) part of the entropy flow comes from the l.h.s. of the kin. eq.:

$$s^{\mu}_{\rm loc} = \mathrm{Tr}\sum_a \int \frac{d^4p}{(2\pi)^4}\left[\left(v^{\mu} - \frac{\partial \mathrm{Re}\Pi^R_a}{\partial p_{\mu}}\right)\left(\mp \widetilde{F}_a \ln\frac{\widetilde{F}_a}{A_a} - F_a \ln\frac{F_a}{A_a}\right)\right.$$
$$\left. - \mathrm{Re}G^R_a \left(\mp \ln\frac{\widetilde{F}_a}{A_a}\frac{\partial}{\partial p_{\mu}}\left(\Gamma_a \frac{\widetilde{F}_a}{A_a}\right) - \ln\frac{F_a}{A_a}\frac{\partial}{\partial p_{\mu}}\left(\Gamma_a\frac{F_a}{A_a}\right)\right)\right]$$

Restored here summation in "a" illustrates that, although this entropy expres. accounts for interactions among all particles, the entropy can be expressed as a sum of the individual contributions, each determined by its $\mathrm{Re}\Pi^R_a$ and Γ_a.

In BM choice the entropy flow can be presented in a more transparent form

$$s^{\mu}_{\rm loc} = \mathrm{Tr}\sum_a \int \frac{d^4p}{(2\pi)^4} A^{\mu}_{sa}(X,p)\sigma_a(X,p), \quad \sigma_a(X,p) = \mp[1\mp f]\ln[1\mp f] - f\ln f,$$

where $A^{\mu}_{sa}(X,p) = \frac{A_a\Gamma_a}{2}B^{\mu}_a$ has the meaning of an entropy-flow spectral function, while $B^{\mu}_a = A_a\left[\left(v^{\mu} - \frac{\partial \mathrm{Re}\Pi^R}{\partial p_{\mu}}\right) - M_a\Gamma_a^{-1}\frac{\partial \Gamma_a}{\partial p_{\mu}}\right]$, is the flow spectral function with B^0 function satisfying the sum rule $\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}B^0_a = 1$. For $\Gamma_a \to 0$ we restore correct qp limit.

Markovian entropy flow and H theorem in KB choice

In KB choice

$$\partial_\mu s^\mu_{\rm loc}(X) = \mathrm{Tr}\sum_a \int \frac{d^4p}{(2\pi)^4} \ln\frac{\widetilde{F}_a}{F_a} C_a(X,p) - \delta H_{KB},$$

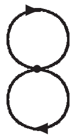
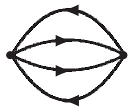
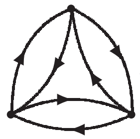
$$s^\mu_{\rm loc} = \mathrm{Tr}\sum_a \int \frac{d^4p}{(2\pi)^4}\left[\left(v^\mu - \frac{\partial \mathrm{Re}\Pi^R_a}{\partial p_\mu}\right)\left(\mp \widetilde{F}_a \ln\frac{\widetilde{F}_a}{A_a} - F_a \ln\frac{F_a}{A_a}\right) - \mathrm{Re}G^R_a\left(\mp \ln\frac{\widetilde{F}_a}{A_a}\frac{\partial}{\partial p_\mu}\left(\Gamma^{out}_a\right) - \ln\frac{F_a}{A_a}\frac{\partial}{\partial p_\mu}\left(\Gamma^{in}_a\right)\right)\right]$$

with $\delta H = -\frac{d^4p}{(2\pi)^4}\mathrm{Re}G^R\{\ln\frac{\widetilde{F}}{F}, \frac{C^{loc}}{A}\}$ being not a full derivative and not a sign definite in general case. Nevertheless we are able to prove H theorem assuming system to be close to local equilibrium. Then $F(X,p) = F_{loc.eq.}(X,p) + \delta F(X,p)$ and we get $\partial_\mu(s^\mu + \delta s^\mu_{KB}) \geq 0$ with $\delta s^\mu_{KB} = \mathrm{Tr}\sum_a \int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{Re}G^R}{A}\frac{\partial \ln(\widetilde{F}/F)}{\partial p_\mu} C^{loc} \propto O(\delta F \partial_X \delta F) + O(\delta F \partial_X F_{loc.eq.})$ that has additional grad. smallness.

Memory effects in entropy flow

If Φ contains more than 2 vertices gradient corrections to the collision term (i.e. C_{mem}) give rise to extra memory contributions to the entropy flow.

Consider example: Φ with 3 vertices:

$$\mathrm{i}\Phi = \mathrm{i}\left(\Phi^{(1)} + \Phi^{(2)} + \Phi^{(3)}\right) = \frac{1}{2}$$  $$+ \frac{1}{4}$$  $$+ \frac{1}{6}$$  . (4)

Proceeding similarly to Markovian case (and assuming positiveness of rates) we obtain

$$\partial_\mu \left(s^\mu_{\text{loc}} + s^\mu_{\text{mem}}\right) \simeq \mathrm{Tr} \int \frac{d^4p}{(2\pi)^4} \ln \frac{\widetilde{F}}{F} (C^{(2)} + C^{(3)}_{\text{loc}}) \geq 0,$$

which is the H-theorem for the non-Markovian kinetic equation under consideration with $s^\mu_{\text{loc}} + s^\mu_{\text{mem}}$ as the proper entropy flow:

$$\begin{aligned} s^\nu_{\text{mem}}(X) \; &= \; -\frac{\mathrm{i}}{2} \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \widetilde{L}^{-+} \left(X; p + \frac{p'}{2}, p - \frac{p'}{2}\right) L^{+-}(X, p') \\ &\times \; \left[\ln \frac{\widetilde{L}^{+-}(X; p + p'/2, p - p'/2)}{\widetilde{L}^{-+}(X; p + p'/2, p - p'/2)} - 1\right] \frac{\partial L^{-+}(X, p')}{\partial p'_\nu}. \end{aligned}$$

Thermodynamic limit of entropy

In equilibrium, knowing Ω one constructs entropy $S = -\left(\partial\Omega/\partial T\right)_\mu/V$, V is the system volume. Thus one finds

$$S = -\mathrm{Tr}\int \frac{d^4p}{(2\pi)^4}\frac{\partial n(\varepsilon)}{\partial T}\left[-2\mathrm{Im}\ \ln\left[-G^R(p_0+\mathrm{i}0,\boldsymbol{p})\right] - \mathrm{Re}G^R\Gamma - A\mathrm{Re}\Pi^R\right] - \frac{\partial\Phi_T}{\partial T}$$

In this expression we separate "local" and "memory" terms $S = S_{\text{loc}} + S_{\text{mem}}$:

$$\begin{aligned}
S_{\text{loc}} &= -\mathrm{Tr}\int \frac{d^4p}{(2\pi)^4}\sigma(\varepsilon)\frac{\partial}{\partial p_0}\left[-2\mathrm{Im}\ \ln[-G^R(p_0+\mathrm{i}0,\boldsymbol{p})] - \mathrm{Re}G^R\Gamma\right],\\
S_{\text{mem}} &= -\frac{\partial\Phi}{\partial T} + \mathrm{Tr}\int \frac{d^4p}{(2\pi)^4}\sigma(\varepsilon)\frac{\partial\left(A\mathrm{Re}\Pi^R\right)}{\partial p_0},\\
\sigma(\varepsilon) &= \mp(1\mp n(\varepsilon))\ln(1\mp n(\varepsilon)) - n(\varepsilon)\ln n(\varepsilon).
\end{aligned}$$

since we reproduce them from our kin. quantities in equilibrium limit.

We may demonstrate that at low T, $S_{mem} \sim T^3\ln T$ gives the famous correction to the specific heat of liquid ^{3}He being quite comparable (numerically) to the leading term in the specific heat ($\sim T$). Thus liquid ^{3}He is a liquid with quite strong memory effects from the point of view of kinetics.

Conclusions

- ✔ A self-consistent non-equilibrium quantum field theory description of macro- systems and first order gradient expansion are constructed. For approximate description to keep self-consistency is very important.
- ✔ Use of Φ derivable approx. garanties conservation laws and thermod. consistency of approx. theory formulated in terms of full Green functions and mean fields.

  Consistency conditions! Expressions for Noether current and energy-momentum tensor in terms of Green funtc. and self-energies.
- ✔ Using Wigner transform. we performed a consistent grad. expansion and derived kin. eq. + algebraic eq. for spectral funct. Other eqs are identically resolved in BM choice. In KB choice they are compatible with those two eqs up to higher grad-s.
- ✔ In consistent grad. expansion scheme memory contributions should be incorporated in collision term
- ✔ In KB choice Noether 4-current and energy-momentum are exactly conserved, consistency conditions are exactly fulfilled (!)
- ✔ We derived quasipart. limit without use of any ansatze.

Conclusions

- ✔ Current contains a drag, back, and memory contrib-s which are compensated producing Noether current.
- ✔ We derived kin. entropy flow and proved H theorem in specific cases.
- ✔ From our kin. entropy we derived thermodynamic limit expression that follows from von Neuman expression.
- ✔ Memory effects in the entropy are responsible for the $\sim T^3 \ln T$ famous correction to the specific heat of liquid ^{3}He .

Some unsolved problems

- ✔ Problems for vector (gauge) fields: (i) Ward identities are not fulfilled since one deals with bare vertices, (ii) No Goldstone theorem. ➡ need modifications for gauge theories.
- ✔ How to construct theory including initial higher order correlations?
- ✔ Is there a unique expression for non-eq. entropy?

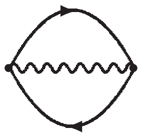
  Does H theorem hold in general case?

  Does positiveness of rates and detailed balance eqs hold in general case and, if not, how does it affect driving the system to equilibrium?
- ✔ To develop a numerical code that would allow to apply this kin. scheme in realistic treatment of h.i.c.
- ✔ etc, etc.

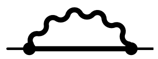
  We are still in the beginning of a long way !!!

Simple example

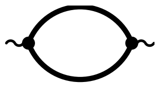
Let us consider a light boson – heavy fermion system taking the simplest approximation for the corresponding Φ functional

$$\mathrm{i}\Phi = \frac{1}{2} \tag{5}$$


in terms of full Green functions of fermions (the bold solid lines) and bosons (the bold wavy line). In this approximation, the fermion self-energy takes the form

$$-\mathrm{i}\Pi_f = \frac{\delta \mathrm{i}\Phi}{\delta \mathrm{i} G_f} = \tag{6}$$


the boson self-energy is given by

$$-\mathrm{i}\left(\frac{1}{2}\right)_{n.b.} \Pi_b = \frac{\delta \mathrm{i}\Phi}{\delta \mathrm{i} G_b} = \tag{7}$$


and Green functions are defined by the standard Dyson's equations.

H theorem

In this case, there are two coupled kin. eqs for fermions and for bosons, leading to

$$\partial_\mu s^\mu_{\text{loc}} = \partial_\mu s^\mu_f + \partial_\mu s^\mu_b = \int \frac{d^4p}{(2\pi)^4} \left(d \ln \frac{\widetilde{F}_f}{F_f} C_f + \frac{1}{2} \ln \frac{\widetilde{F}_b}{F_b} C_b \right), \tag{8}$$

where s^μ_{loc} is given by the sum of the proper fermion (s^μ_f) and boson (s^μ_b) contributions. The collision term can be cast in the form

$$\begin{aligned} \partial_\mu s^\mu_{\text{loc}} \; &= \; d \int \frac{d^4p_{f1}}{(2\pi)^4} \frac{d^4p_{f2}}{(2\pi)^4} \frac{d^4p_b}{(2\pi)^4} g^2 (2\pi)^4 \delta^4 \left(p_{f1} - p_{f2} - p_b\right) \\ &\times \ln \frac{\widetilde{F}_1 F_2 F_b}{F_{\text{f}1} \widetilde{F}_{f2} \widetilde{F}_b} \left(\widetilde{F}_{f1} F_{f2} F_b - F_{f1} \widetilde{F}_{f2} \widetilde{F}_b \right) \geq 0, \end{aligned} \tag{9}$$

that proves H theorem for the given Φ.

Pion cond. phase transition in dense nucleon system

Consider relaxation of a pion distribution in a dense isotopically symmetric (quasi)equilibrium nucleon matter. Assume $\rho_\pi \ll \rho_N$. Using also that $m_\pi/m_N \sim 1/7 \ll 1$ we can neglect a feedback of pions onto nucleons. Thus we may drop pion distrib. dependence in all self-energy terms.

Pion distribution is found from the kin. eq. that in BM choice renders:

$$\frac{\Gamma_\pi}{2} B^\pi_\mu \partial^\mu_x f_\pi(\epsilon, \boldsymbol{p}, t, \boldsymbol{r}) = \Gamma^\pi_{in} - \Gamma^\pi f_\pi,$$

$B^\mu_\pi = A_\pi \left[\left(v^\mu - \frac{\partial \mathrm{Re}\Pi^R_\pi}{\partial p_\mu}\right) - M_\pi \Gamma^{-1}_\pi \frac{\partial \Gamma_\pi}{\partial p_\mu}\right]$. Assuming $f_\pi(\epsilon, \boldsymbol{p}, t, \boldsymbol{r}) = f^{eq}_\pi(\epsilon) + \delta f_\pi(\epsilon, \boldsymbol{p}, t, \boldsymbol{r})$ we find

$$\frac{1}{2} B^\mu_\pi \partial^\mu_x \delta f_\pi + \delta f_\pi = 0,$$

whereas for space-homogeneous case it follows the solution:

$$\delta f_\pi(\epsilon, \boldsymbol{p}) = \delta f_0(\epsilon, \boldsymbol{p}) e^{-2t/B_0(\epsilon, \boldsymbol{p})}$$

.

Pion cond. phase transition in a dense nucleon system

Pion has attractive p wave interaction to nucleon. Let ρ_N being sufficiently high. Since $\mathrm{Re}\Pi_\pi^R \propto -\boldsymbol{p}^2 p_F^N(\rho_N)$ the quantity

$$\widetilde{\omega}^2 \equiv -\mathrm{min}_{|\boldsymbol{p}|}\mathrm{Re}G_\pi^R(\epsilon = 0, |\boldsymbol{p}|) = m_\pi^2 + p_m^2 + \mathrm{Re}\Pi_\pi^R(\epsilon = 0, p_m)$$

changes sign at some critical density $\rho_N = \rho_{crit}$ and becomes negative that signalizes pion condensation phase transition. For small ϵ, and $|\boldsymbol{p}| \simeq p_m$ the pion width $\Gamma_\pi = \beta(p_m)\epsilon$, with $\beta(p_m) = const > 0$, and therefore the second term is dominant in B_0^π.
This case is not described by qp. approx. (!).

Thus $B_\pi^0 = \frac{2\beta}{\widetilde{\omega}^2}$ and

$$\delta f_\pi(\epsilon, \boldsymbol{p}) \simeq \delta f_0 \mathrm{exp}(-\widetilde{\omega}^2 t/\beta).$$

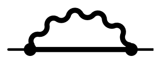
Thus initial virtual pion fluctuation (for small ϵ and $\boldsymbol{p} \neq 0$) is damped for $\rho_N < \rho_{crit}$ (when $\widetilde{\omega}^2 > 0$) and it grows in time for $\rho_N > \rho_{crit}$ (when $\widetilde{\omega}^2 < 0$).

- ✔ In case $\widetilde{\omega}^2 < 0$, we have no source term in kin. eq. generating an initial pion distribution since we have suppressed initial correl. which generate an accident force. The corresponding term is usually added in phenomenological treatment of the dynamics of phase transitions.
- ✔ In this particular example the pion width drives the phase transition.

Hot hadron vacuum

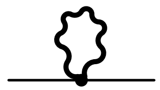
Consider hot pion-nucleon vacuum ($\mu_N \simeq 0$). For temperature $m_\pi \lesssim T \ll m_N$, typical nucleon momenta $\sqrt{2m_N T}$ and typical pion momenta are $\sim T \ll \sqrt{2m_N T}$.
➡ soft thermal loop (STL) approx.: drop pion momentum in nucleon Green func. in

$$-\mathrm{i}\Pi_f = \frac{\delta \mathrm{i}\Phi}{\delta \mathrm{i}G_f} = \tag{10}$$



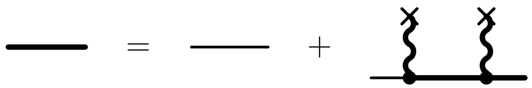
Then nucleon Green function decouples and self-energy reduces to the pion tadpole

$$J = \qquad \simeq g^2_{p.v.}\frac{T^4}{m_\pi^2}, \tag{11}$$



$g_{p.v.} \simeq 1$ is dimensionless πN coupling constant.

Thus Dyson equation for the nucleon gets clear diagrammatic interpretation:



(12)

describing the fermion propagation in an external field ϕ_c^{ext}, $\phi_c^{\mathrm{ext}} \cdot (\phi_c^{\mathrm{ext}})^* = J/g^2_{p.v.}$. Nucleon undergoes multiple quasi-elastic rescatterings on pairs of quasi-static pion impurities. The value J is proportional to the density of impurities. Thus, it demonstrates the intensity of the multiple elastic scattering.

Hot hadron vacuum

To suppress spin structure assume for simplicity that nucleons are non-rel. part. Then nucleon Dyson eq. yields

$$G_N^R = G_N^{0R} + G_N^{0R} J (G_N^R)^2$$

with $G_N^{0R} = 1/(\epsilon - \epsilon_p)$, $\epsilon_p = \frac{\boldsymbol{p}^2}{2m_N}$. This algebraic eq. is easily resolved:

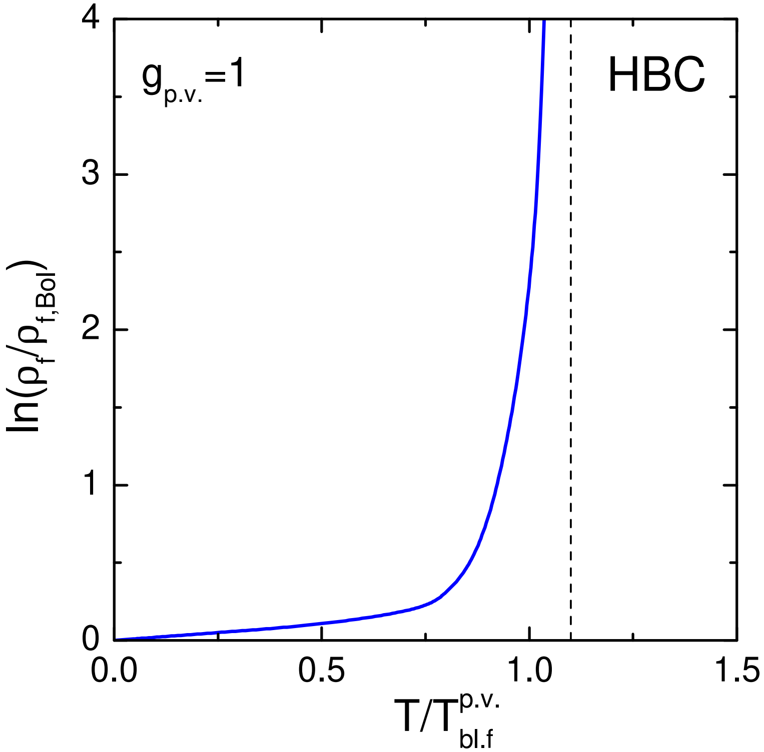
$$G^R = \frac{(\epsilon - \epsilon_p)}{2J} \pm \sqrt{\left(\frac{(\epsilon - \epsilon_p)}{2J}\right)^2 - \frac{1}{J}}$$

where we should take $''-''$ getting appropriate qp. limit for $|\epsilon - \epsilon_p| \sim T \gg J$. In the temperature regime $m_\pi \lesssim T \ll m_N$ we have opposite limit and the nucleon Green function losses its qp. pole structure (!) acquiring a broad width ($\mathrm{Im} G^R \neq 0$) $\Rightarrow$ a blurring of the nucleon continuum. Nucleon momentum distrib.:

$$n_p = -4 \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \frac{2\mathrm{Im} G^R}{e^{(m_N+\epsilon)/T} + 1} \sim n_p^{Boltz.} e^{2\sqrt{J}/T} \left(\frac{T}{\sqrt{J}}\right)^{3/2} \propto T^3 e^{-m^*(T)/T}$$

For $T \simeq T^{p.v}_{\mathrm{bl.f}}$, $2\sqrt{J}(T^{p.v}_{\mathrm{bl.f}}) \simeq m_N$, the effective nucleon mass $m^*(T) \to 0$. One can easily estimate that $T^{p.v}_{\mathrm{bl.f}} \sim 140 - 200$ MeV (not as large as m_N (!)).

Hot hadron vacuum



Logarithm of the ratio of the fermion density to the corresponding Boltzmann quantity for $g_{p.v} = 1$ as function of the temperature in units of $T^{p.v}_{\mathrm{bl.f}}$,

cf. Urbah law; multiple electron-phonon interact. in doped semiconductors.

The vertical dash-line indicates the area of a possible hot boson condensation (HBC), for details see D.V. Nucl. Phys. **A 744** (2004) 378