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HEAVY QUARK PHYSICS
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Laurent series expansion of massive scalar one-loop integrals to $\mathcal{O}(\varepsilon^2)$

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- Physics motivation: NNLO $\mathcal{O}(\alpha_s^4)$ corrections to heavy hadron production

- Aim of this project: One-loop contributions to

$$i) \quad g + g \to Q + \bar{Q} \qquad\qquad ii) \quad q + \bar{q} \to Q + \bar{Q}$$

up to $\mathcal{O}(\varepsilon^2)$. They are needed for the loop-by-loop $\mathcal{O}(\alpha_s^4)$ NNLO contributions to heavy hadron production.

Necessity of NNLO calculation for hadronic heavy hadron production

- Theoretical errors due to possible choices of renormalization and factorization scales are expected to be greatly reduced at NNLO QCD.

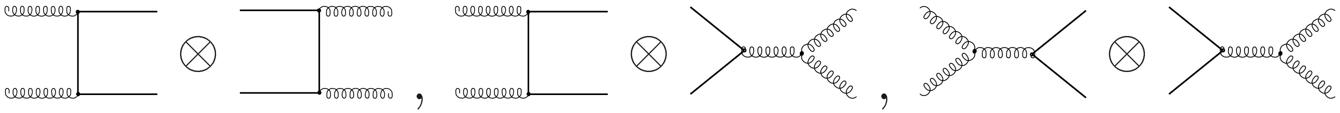
- Hope: better agreement between theory and experiment at NNLO level.

Ingredients for NNLO calculations

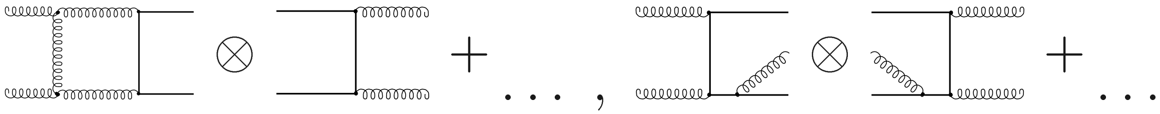
$$d\sigma \sim |M_{fi}|^2$$

- History:

LO QCD for heavy hadron production $\sim \alpha_s^2$ (Glück *et al.*, 1978, ...):

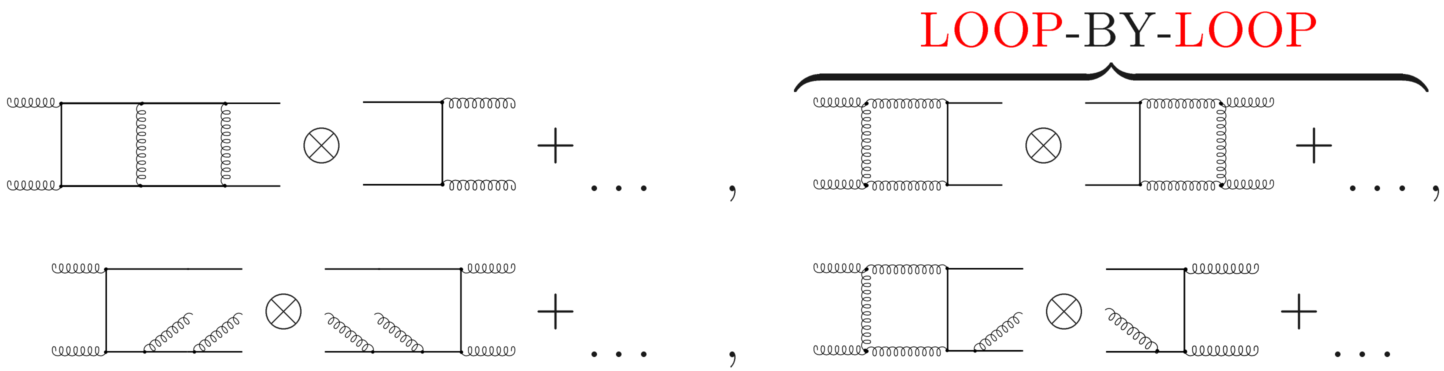


NLO QCD $\sim \alpha_s^3$ (S. Dawson *et al.*, '88, Beenakker *et al.*, '89, ...) :



NNLO QCD $\sim {\alpha_s}^4$ (pending):

Generic diagrames :



- All necessary scalar one-, two-, three- and four-point functions for LOOP-BY-LOOP part of NNLO QCD **have been computed** up to $\mathcal{O}(\varepsilon^2)$.

List of scalar n–point functions up to $\mathcal{O}(\varepsilon^2)$:

	Nomenclature of Beenakker	Our nomenclature	Novelty	Comments
1-point	$A(m)$	A	–	Re
2-point	$B(p_4 - p_2, 0, m)$	B_1	–	Re
	$B(p_3 + p_4, m, m)$	B_2	–	Re, Im
	$B(p_4, 0, m)$	B_3	–	Re
	$B(p_2, m, m)$	B_4	–	Re
	$B(p_3 + p_4, 0, 0)$	B_5	–	Re, Im
	$B(p_3 + p_4, 0, 0)$	B_5	–	Re, Im
3-point	$C(p_4, p_3, 0, m, 0)$	C_1	new	Re, Im
	$C(p_4, -p_2, 0, m, m)$	C_2	new	Re
	$C(-p_2, p_4, 0, 0, m)$	C_3	–	Re
	$C(-p_2, -p_1, 0, 0, 0)$	C_4	–	Re, Im
	$C(-p_2, -p_1, m, m, m)$	C_5	–	Re, Im
	$C(p_3, p_4, m, 0, m)$	C_6	–	Re, Im
4-point	$D(p_4, -p_2, -p_1, 0, m, m, m)$	D_1	new	Re, Im
	$D(-p_2, p_4, p_3, 0, 0, m, 0)$	D_2	new	Re, Im
	$D(-p_2, p_4, -p_1, 0, 0, m, m)$	D_3	new	Re

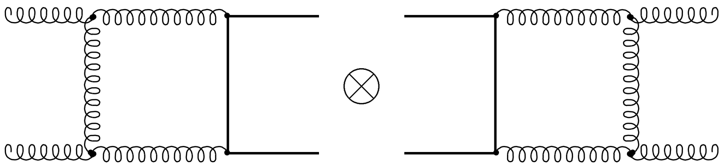
Results were checked

- numerically step by step
- we agree with the numerical results provided by M.M. Weber who developed an all-purpose method to calculate general one-loop results
  (M.M. Weber, Acta Phys. Polon. **B35**, 2655 (2004). ArXiv: hep-ph/0410166)
- the evaluation of our results is faster by orders of magnitude (in particular for imaginary parts)
- the evaluation time is quite uniform over phase space whereas Weber's is not uniform over phase space

Calculation of four-point scalar function

- The term LOOP-BY-LOOP

$$(\underbrace{\frac{a_{-2}}{\varepsilon^2}+\frac{a_{-1}}{\varepsilon}+a_0}_{known}+\underbrace{a_1\varepsilon+a_2\varepsilon^2}_{new}+\dots)(\frac{a_{-2}}{\varepsilon^2}+\frac{a_{-1}}{\varepsilon}+a_0+a_1\varepsilon+a_2\varepsilon^2+\dots)^*$$



$\mathcal{O}(\varepsilon^2)$ terms come from

i) spin algebra $g_{\mu\nu}g^{\mu\nu} = n = 4-2\,\varepsilon$

ii) Passarino-Veltman decomposition of tensor integrals $g_{\mu\nu}g^{\mu\nu} = n = 4-2\,\varepsilon$

iii) scalar master loop integrals

$$D(q_1, q_2, q_3, m_1, m_2, m_3, m_4) =$$
$$\mu^{2\varepsilon} \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - m_1^2 + i\delta)[(q+q_1)^2 - m_2^2 + i\delta]} \times$$
$$\times \frac{1}{[(q+q_1+q_2)^2 - m_3^2 + i\delta][(q+q_1+q_2+q_3)^2 - m_4^2 + i\delta]}$$

μ - arbitrary mass scale, m_k - internal masses,

$n = 4 - 2\,\varepsilon$ - dimension of space-time.

Applying Feynman parameterization we obtain

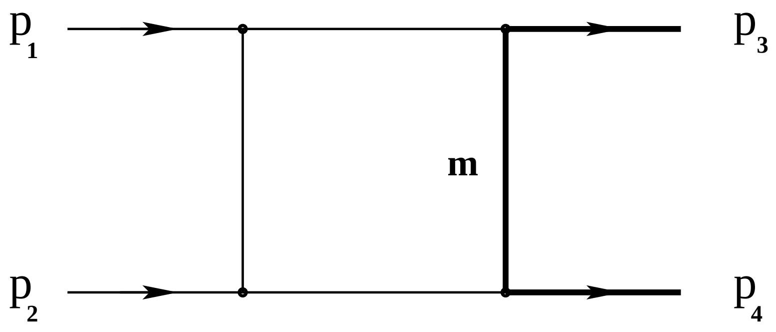
$$D = iC_\varepsilon(m^2)(1+\varepsilon)(m^2)^{-2} \int_0^1 dx_1 dx_2 dx_3\; x_1^2 x_2 K^{-2-\varepsilon}$$

where $C_\varepsilon(m^2) \equiv \frac{\Gamma(1+\varepsilon)}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^\varepsilon$ and the kernel K given by

$$\begin{aligned} m^2 K \quad = \quad & -abq_1^2 - ac(q_1+q_2)^2 - ad(q_1+q_2+q_3)^2 - bcq_2^2 \\ & -bd(q_2+q_3)^2 - cdq_3^2 + am_1^2 + bm_2^2 + cm_3^2 + dm_4^2 - i\delta. \end{aligned}$$

The set $\{a, b, c, d\}$ is an arbitrary choice of permutations of
$\{x_1 x_2 x_3,\; x_1 x_2 (1 - x_3),\; x_1 (1 - x_2),\; 1 - x_1\}$.

- One of the three four-point scalar master integrals



$$D_2 \equiv D(-p_2, p_4, p_3, 0, 0, m, 0),$$
$$K = ac\tilde{t} - bd\tilde{s} + c^2 - i\delta.$$

$$s \equiv (p_1 + p_2)^2,\ t \equiv T - m^2 \equiv (p_1 - p_3)^2 - m^2$$
$$\tilde{s} \equiv s/m^2,\ \tilde{t} \equiv -t/m^2.$$

The kinematic conditions $(g + g \to Q + \bar{Q})$:

$$s \geq 4m^2 \quad , \quad t_{min}(s) \leq t(s) \leq t_{max}(s),$$
$$t_{max}(s) = -s/2\left(1 - \sqrt{1 - 4m^2/s}\right) \quad , \quad t_{min}(s) = -s/2\left(1 + \sqrt{1 - 4m^2/s}\right).$$

- Because of **IR/M** divergences there exist terms $\sim \frac{1}{\varepsilon^2} \implies$

Calculations for NNLO BOX **must be performed** up to terms of order ε^2 !

Subtraction terms

- Consider

$$\int_0^1 dx\, x^{-1+\varepsilon} f(x,\varepsilon) \quad \text{with}$$

  $f(x,\varepsilon)$ - integrable in $[0,1]$
  $f(x,\varepsilon):\quad \forall\ \ \varepsilon$ in the vicinity of 0 $\ \exists\ \ \frac{\partial^n f(x,\varepsilon)}{\partial\varepsilon^n} \quad \Rightarrow$
  $f(x,\varepsilon) = f^{(0)}(x) + \varepsilon f^{(1)}(x) + \ldots\ \ .$

- Expansion
  $x^{-1+\varepsilon} f(x,\varepsilon) = \frac{1}{x} f^{(0)}(x) + \varepsilon \left(\frac{\ln x}{x} f^{(0)}(x) + \frac{1}{x} f^{(1)}(x) \right) + \ldots$
  This does not solve a problem!

- Using the "plus" prescription

$$\int_0^1 dx\, \left(x^{-1+\varepsilon}\right)_+ f(x,\varepsilon) = \int_0^1 dx x^{-1+\varepsilon} (f(x,\varepsilon) - f(0,\varepsilon)),$$

  we make the r.h.s. of the following identity integrable,

$$\int_0^1 dx x^{-1+\varepsilon} f(x,\varepsilon) = \int_0^1 dx\, \left(x^{-1+\varepsilon}\right)_+ f(x,\varepsilon) + f(0,\varepsilon) \int_0^1 dx x^{-1+\varepsilon}.$$

- D_2 up to ε^0 is

$$\frac{iC_\varepsilon(m^2)}{st}\Big[\frac{2}{\varepsilon^2} - \frac{\ln\frac{s}{m^2} + 2\,\ln\frac{-t}{m^2}}{\varepsilon} + 2\,\ln\frac{s}{m^2}\,\ln\frac{-t}{m^2} - 5\,\zeta(2) + i(\frac{\pi}{\varepsilon} - 2\pi\ln\frac{-t}{m^2})\Big].$$

Recalculated and agree with results of

$\mapsto$W.J.P. Beenakker *et al.*, Phys. Rev. **D 40** (1989) 54 (real part)

$\mapsto$ I. Bojak and M. Stratmann, hep-ph/0005120 (real part)

$\mapsto$ W.Bernreuther *et al.*, Phys. Lett. **B 368** (1996) 153 (imaginary part)

- for the scalar $(0,0,m,0)$-BOX up to order ε^1 we obtain:

Real part $\Rightarrow \frac{iC_\varepsilon (m^2)}{12st}\varepsilon\Big[6\ \ln^3 \frac{s}{m^2} + 20\ \ln^3 \frac{-t}{m^2} + \ln^3 x +$

$6\ \ln^2 x\ \ln \frac{-(s+2\,t-s\,\beta)}{2\,m^2} - 12\ \ln x\ \ln^2 \frac{-(s+2\,t-s\,\beta)}{2\,m^2}) + 8\ \ln^3 \frac{-(s+2\,t-s\,\beta)}{2\,m^2} -$

$12\ \ln^2 \frac{-t}{m^2}\ \left(2\ \ln(-1-\frac{t}{m^2}) + 3\ \ln x - \ln \frac{-(s+2\,t-s\,\beta)}{2\,m^2} + 4\ \ln \frac{s+2\,t+s\,\beta}{2\,m^2}\right)$

$-\ 3\ \ln^2 \frac{s}{m^2}\ \left(6\ \ln \frac{-t}{m^2} + 3\ \ln x + 2\ \ln \frac{-(s+2\,t-s\,\beta)}{2\,m^2} + 4\ \ln \frac{s+2\,t+s\,\beta}{2\,m^2}\right) +$

$12\ \ln \frac{s}{m^2} + 24\ \ln \frac{-t}{m^2} + 24\ \ln \frac{s}{m^2}\ \ln \frac{-t}{m^2} + 48\,Li_3(\frac{m^2+t}{t}) + 48\,Li_3(1+\frac{t}{m^2}) +$

$24\,Li_3(-x) - 24\,Li_3(\frac{-2\,m^2+t\,(-1+\beta)}{t\,(1+\beta)}) - 48\,Li_3(\frac{2\,m^2+t-t\,\beta}{2\,m^2}) +$

$24\,Li_3(\frac{2\,m^2+t-t\,\beta}{m^2\,(1+\beta)}) - 24\,Li_3(\frac{2\,m^2+t-t\,\beta}{t-t\,\beta}) - 48\,Li_3(\frac{2\,m^2}{2\,m^2+t+t\,\beta}) +$

$24\,Li_3(-\frac{m^2\,(-1+\beta)}{2\,m^2+t+t\,\beta}) - 24\,Li_3(\frac{t\,(-1+\beta)}{2\,m^2+t+t\,\beta}) - 24\,Li_3(\frac{2\,m^2+t+t\,\beta}{t+t\,\beta}) -$

$6\ \ln \frac{-t}{m^2}\ \Big(4 + 2\ \ln^2(-1-\frac{t}{m^2}) - \ln^2 x - 4\ \ln x\ \ln \frac{-(s+2\,t-s\,\beta)}{2\,m^2} +$

$6\ \ln^2 \frac{-(s+2\,t-s\,\beta)}{2\,m^2} - 4\ \ln(-1-\frac{t}{m^2})\ \left(\ln x + \ln \frac{-(s+2\,t-s\,\beta)}{2\,m^2} + \ln \frac{s+2\,t+s\,\beta}{2\,m^2}\right) +$

$8\,Li_2(1+\frac{t}{m^2}) + 8\,Li_2(\frac{t\,(-1+\beta)}{2\,m^2}) + 4\,Li_2(\frac{m^2\,(-1+\beta)}{\left(m^2+t\right)\,(1+\beta)}) -$

$4\,Li_2(\frac{\left(m^2+t\right)\,(-1+\beta)}{m^2\,(1+\ \beta)}) + 8\,Li_2(\frac{2\,m^2}{2\,m^2+t+t\,\beta}) - 12\,\zeta(2)\Big) - 12\ \ln x\,\zeta(2) +$

$$48\ \ln\frac{-(s+2\,t-s\,\beta)}{2\,m^2}\,\zeta(2)-6\ \ln\frac{s}{m^2}\,\Big(2+4\ \ln^2\frac{-t}{m^2}-\ln^2 x-$$
$$2\ \ln x\ \ln\frac{s+2\,t+s\,\beta}{2\,m^2}-4\ \ln\frac{-t}{m^2}\,(-1+\ln x+\ln\frac{-(s+2\,t-s\,\beta)}{2\,m^2}+2\ \ln\frac{s+2\,t+s\,\beta}{2\,m^2})-$$
$$4\,Li_2(\frac{t\,(-1+\beta)}{2\,m^2})-4\,Li_2(\frac{2\,m^2}{2\,m^2+t+t\,\beta})+2\,\zeta(2)\Big)-36\,\zeta(3)\Big]$$

$\color{blue}{Imaginary}$ (absorptive) part $\Rightarrow$

$$-\frac{iC_\varepsilon(m^2)}{4st}\varepsilon\pi\Big[3\ \ln^2\frac{s}{m^2}-4\ \ln\frac{s}{m^2}\ \ln\frac{-t}{m^2}-8\ \ln^2\frac{-t}{m^2}-2\ \ln\frac{s}{m^2}\ \ln x+$$
$$4\ \ln\frac{-t}{m^2}\ \ln x+3\ \ln^2 x-4\ \ln\frac{s}{m^2}\ \ln\frac{-(s+2\,t-s\,\beta)}{2\,m^2}-4\ \ln x\ \ln\frac{-(s+2\,t-s\,\beta)}{2\,m^2}+$$
$$4\ \ln^2\frac{-(s+2\,t-s\,\beta)}{2\,m^2}-4\ \ln\frac{s}{m^2}\ \ln\frac{s+2\,t+s\,\beta}{2\,m^2}+8\ \ln\frac{-t}{m^2}\ \ln\frac{s+2\,t+s\,\beta}{2\,m^2}+$$
$$4\ \ln x\ \ln\frac{s+2\,t+s\,\beta}{2\,m^2}+8\,Li_2(\frac{t\,(-1+\beta)}{2\,m^2})+8\,Li_2(\frac{2\,m^2}{2\,m^2+t+t\,\beta})+8\,\zeta(2)\Big],$$

where $\beta=\sqrt{1-4m^2/s}$ and $x=(1-\beta)/(1+\beta)$.

▲ Imaginary part also needed ! ($|A|^2=(ReA)^2+(ImA)^2$)

The procedure of analytical integration and the ensuing simplification for $\mathcal{O}(\varepsilon^1)$ is very complex. The answer includes classical polylogarithms:

$$Li_2(x) = -\int_0^1 \frac{\ln(1-xy)}{y}dy \ , \ Li_3(x) = \int_0^1 \frac{Li_2(xy)}{y}dy$$

■ As an illustration: untreated answer for ε^1–contribution after integration fills about 30 pages of computer output and includes about 80 different Li_2 and about 65 different Li_3 .

■ The hard work is to simplify the expression using suitable identities for Li_2 and Li_3. *Semi-automatic computer codes* have been written for these purposes.

Algorithm for the simplification of Li_2 and Li_3.

Lewin " Polylogarithms and associated functions" $\Rightarrow$

$$
\begin{aligned}
Li_2(z) &= \zeta(2) - \ln(z)\ln(1-z) - Li_2(1-z), z \in C,\ z \notin (-\infty, 0] \cup [1, +\infty)\,, \\
Li_2(z) &= -\zeta(2) - 1/2\ln^2(-z) - Li_2(1/z), z \in C,\ z \notin [0, +\infty)\,.
\end{aligned}
$$

Relations between:

$$
Li_2(z), Li_2(1-z), Li_2\left(\tfrac{1}{z}\right), Li_2\left(\frac{1}{1-z}\right), Li_2\left(\tfrac{z-1}{z}\right), Li_2\left(\frac{z}{z-1}\right).
$$

In addition:

$$
\begin{aligned}
Li_2(z) + Li_2(-z) &= 1/2 Li_2(z^2),\ z \in C, z \notin (-\infty, -1] \cup [1, +\infty) \\
\text{Hill}: \qquad Li_2(xy) &= Li_2(x) + Li_2(y) - Li_2\left(\frac{x(1-y)}{1-xy}\right) \\
-Li_2\left(\frac{y(1-x)}{1-xy}\right) &- \ln\left(\frac{1-x}{1-xy}\right)\ln\left(\frac{1-y}{1-xy}\right)
\end{aligned}
$$

For Hill : n -number of Li_2's $\rightarrow$ $n(n-1)/2$ pairs to consider, for each pair $6 \times 6 \times 1/2 \times 3 \times (n-2) \times 6 = 324 \times (n-2)$ comparisons

$$
\begin{aligned}
Li_3(z) &= -\zeta(2)\ln(-z) - 1/6\ln^3(-z) + Li_3(1/z), z \in C,\ z \notin [0, +\infty), \\
Li_3(z) &= \zeta(3) + \zeta(2)\ln(1-z) - Li_3(1-z) - Li_3\left(\frac{z}{z-1}\right), \\
&\quad z \in C,\ z \notin (-\infty, 0] \cup [1, +\infty).
\end{aligned}
$$

Relations between:

$$Li_3(z), Li_3(1-z), Li_3\left(\tfrac{1}{z}\right), Li_3\left(\tfrac{1}{1-z}\right), Li_3\left(\tfrac{z-1}{z}\right), Li_3\left(\tfrac{z}{z-1}\right).$$

In addition:

$$Li_3(z) + Li_3(-z) = 1/4 Li_3(z^2),\ z \in C, z \notin (-\infty, -1] \cup [1, +\infty)$$

- Calculation of ε^2–contributions

▲ $\mathcal{O}(\varepsilon^2)$ result contains functions of type ln, Li_2, Li_3, Li_4, $\zeta(2,3,4)$ and products of these and also new class of functions such as

$$L_{\pm++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \int_0^1 dy \frac{\ln(\alpha_1 \pm y)\ln(\alpha_2 + y)\ln(\alpha_3 + y)}{\alpha_4 + y}$$

and

$$L_{+}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \int_0^1 dy \frac{\ln(\alpha_1 + y)\mathrm{Li}_2(\alpha_2 + \alpha_3 y)}{\alpha_4 + y} \ ,$$

where the α_j's are combinations of kinematic variables.

- The L-functions are fast to evaluate numerically
- They allow one to present the result in a concise manner

The L-functions (and our results) can be expressed in terms of multiple polylogarithms (see later)

Reduction of L-functions to a minimal set

Domain:

$$
\begin{aligned}
L_{+++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &: \quad \alpha_1>0,\alpha_2>0,\alpha_3>0,\alpha_4<-1 \quad \text{or} \quad \alpha_4>0;\\
L_{-++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &: \quad \alpha_1>1,\alpha_2>0,\alpha_3>0,\alpha_4<-1 \quad \text{or} \quad \alpha_4>0;\\
L_{+}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &: \quad \alpha_1>0,\alpha_2\le 1,\alpha_2+\alpha_3\le 1,\alpha_3\neq 0,\alpha_4<-1 \quad \text{or} \quad \alpha_4>0.
\end{aligned}
$$

$$
L_{+++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\begin{cases}
\alpha_4>0: & \ln(\alpha_1+y)\ln(\alpha_2+y)\ln(\alpha_3+y)\ln(\alpha_4+y)\Big|_0^1\\
& -L_{+++}(\alpha_4,\alpha_2,\alpha_3,\alpha_1)-L_{+++}(\alpha_1,\alpha_4,\alpha_3,\alpha_2)\\
& -L_{+++}(\alpha_1,\alpha_2,\alpha_4,\alpha_3);\\
\alpha_4<-1: & \ln(\alpha_1+y)\ln(\alpha_2+y)\ln(\alpha_3+y)\ln(-\alpha_4-y)\Big|_0^1\\
& -L_{-++}(-\alpha_4,\alpha_2,\alpha_3,\alpha_1)-L_{-++}(-\alpha_4,\alpha_1,\alpha_3,\alpha_2)\\
& -L_{-++}(-\alpha_4,\alpha_1,\alpha_2,\alpha_3);
\end{cases}
$$

$$
L_{-++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\begin{cases}
\alpha_4>0: & \ln(\alpha_1-y)\ln(\alpha_2+y)\ln(\alpha_3+y)\ln(\alpha_4+y)\Big|_0^1\\
& -L_{+++}(\alpha_4,\alpha_2,\alpha_3,-\alpha_1)-L_{-++}(\alpha_1,\alpha_4,\alpha_3,\alpha_2)\\
& -L_{-++}(\alpha_1,\alpha_2,\alpha_4,\alpha_3);\\
\alpha_4<-1: & \ln(\alpha_1-y)\ln(\alpha_2+y)\ln(\alpha_3+y)\ln(-\alpha_4-y)\Big|_0^1\\
& -L_{-++}(-\alpha_4,\alpha_2,\alpha_3,-\alpha_1)\\
& +L_{-++}(\alpha_3+1,\alpha_1-1,-\alpha_4-1,-\alpha_2-1)\\
& +L_{-++}(\alpha_2+1,\alpha_1-1,-\alpha_4-1,-\alpha_3-1).
\end{cases}
$$

$$
L_{+}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=
$$

$$
\begin{cases}
\alpha_4 > 0, \alpha_3 > 0: & \ln(\alpha_1 + y)\mathrm{Li}_2(\alpha_2 + \alpha_3 y)\ln(\alpha_4 + y)\Big|_0^1 \\
& -L_+(\alpha_4, \alpha_2, \alpha_3, \alpha_1) \\
& +L_{-++}(\frac{1-\alpha_2}{\alpha_3}, \alpha_1, \alpha_4, \frac{\alpha_2}{\alpha_3}) + \ln(\alpha_3)\int\limits_0^1 \frac{\ln(\alpha_1+y)\ln(\alpha_4+y)}{\frac{\alpha_2}{\alpha_3}+y}dy; \\
\alpha_4 > 0, \alpha_3 < 0: & \ln(\alpha_1 + y)\mathrm{Li}_2(\alpha_2 + \alpha_3 y)\ln(\alpha_4 + y)\Big|_0^1 \\
& -L_+(\alpha_4, \alpha_2, \alpha_3, \alpha_1) \\
& +L_{+++}(\frac{\alpha_2-1}{\alpha_3}, \alpha_1, \alpha_4, \frac{\alpha_2}{\alpha_3}) + \ln(-\alpha_3)\int\limits_0^1 \frac{\ln(\alpha_1+y)\ln(\alpha_4+y)}{\frac{\alpha_2}{\alpha_3}+y}dy; \\
\alpha_4 < -1, \alpha_3 > 0: & \ln(\alpha_1 + y)\mathrm{Li}_2(\alpha_2 + \alpha_3 y)\ln(-\alpha_4 - y)\Big|_0^1 \\
& +L_+(-\alpha_4 - 1, \alpha_2 + \alpha_3, -\alpha_3, -\alpha_1 - 1) \\
& -L_{-++}(\alpha_1 + 1, \frac{1-\alpha_2-\alpha_3}{\alpha_3}, -\alpha_4 - 1, -\frac{\alpha_2+\alpha_3}{\alpha_3}) \\
& + \ln(\alpha_3)\int\limits_0^1 \frac{\ln(\alpha_1+y)\ln(-\alpha_4-y)}{\frac{\alpha_2}{\alpha_3}+y}dy; \\
\alpha_4 < -1, \alpha_3 < 0: & \ln(\alpha_1 + y)\mathrm{Li}_2(\alpha_2 + \alpha_3 y)\ln(-\alpha_4 - y)\Big|_0^1 \\
& +L_+(-\alpha_4 - 1, \alpha_2 + \alpha_3, -\alpha_3, -\alpha_1 - 1) \\
& +L_{-++}(-\alpha_4, \alpha_1, \frac{\alpha_2-1}{\alpha_3}, \frac{\alpha_2}{\alpha_3}) + \ln(-\alpha_3)\int\limits_0^1 \frac{\ln(\alpha_1+y)\ln(-\alpha_4-y)}{\frac{\alpha_2}{\alpha_3}+y}dy,
\end{cases}
$$

$$
\text{additional condition}: \frac{\alpha_2}{\alpha_3} < -1 \quad \text{or} \quad \frac{\alpha_2}{\alpha_3} > 0
$$

$$
L_+(\alpha_1, \alpha_2, \alpha_3, \alpha_4) =
$$

$$\alpha_1 < \alpha_4, \alpha_3 > 0: \quad \mathrm{Li}_2(\alpha_2+\alpha_3 y)\left(\ln(\alpha_1+y)\ln(\frac{\alpha_4+y}{\alpha_4-\alpha_1})+\mathrm{Li}_2(\frac{\alpha_1+y}{\alpha_1-\alpha_4})\right)\Big|_0^1 + L_{-++}(\frac{1-\alpha_2}{\alpha_3},\alpha_1,\alpha_4,\frac{\alpha_2}{\alpha_3})$$
$$-L_+(\frac{1-\alpha_2-\alpha_3}{\alpha_3},\frac{\alpha_1+1}{\alpha_1-\alpha_4},\frac{1}{\alpha_4-\alpha_1},-\frac{\alpha_2+\alpha_3}{\alpha_3})+$$
$$\int_0^1 \frac{\ln(\alpha_3)\left[\mathrm{Li}_2(\frac{\alpha_1+y}{\alpha_1-\alpha_4})+\ln(\alpha_1+y)\ln(\frac{\alpha_4+y}{\alpha_4-\alpha_1})\right]-\ln(\alpha_4-\alpha_1)\ln(\alpha_1+y)\ln(\frac{1-\alpha_2}{\alpha_3}-y)}{\frac{\alpha_2}{\alpha_3}+y}dy;$$

$$\alpha_1 < \alpha_4, \alpha_3 < 0: \quad \mathrm{Li}_2(\alpha_2+\alpha_3 y)\left(\ln(\alpha_1+y)\ln(\frac{\alpha_4+y}{\alpha_4-\alpha_1})+\mathrm{Li}_2(\frac{\alpha_1+y}{\alpha_1-\alpha_4})\right)\Big|_0^1 + L_{+++}(\frac{\alpha_2-1}{\alpha_3},\alpha_1,\alpha_4,\frac{\alpha_2}{\alpha_3})$$
$$+L_+(\frac{\alpha_2-1}{\alpha_3},\frac{\alpha_1}{\alpha_1-\alpha_4},\frac{1}{\alpha_1-\alpha_4},\frac{\alpha_2}{\alpha_3})+$$
$$\int_0^1 \frac{\ln(-\alpha_3)\left[\mathrm{Li}_2(\frac{\alpha_1+y}{\alpha_1-\alpha_4})+\ln(\alpha_1+y)\ln(\frac{\alpha_4+y}{\alpha_4-\alpha_1})\right]-\ln(\alpha_4-\alpha_1)\ln(\alpha_1+y)\ln(\frac{\alpha_2-1}{\alpha_3}+y)}{\frac{\alpha_2}{\alpha_3}+y}dy;$$

$$\alpha_1 > \alpha_4 > 0, \alpha_3 > 0 \quad : \quad \mathrm{Li}_2(\alpha_2+\alpha_3 y)\left(\ln(\alpha_1-\alpha_4)\ln(\alpha_4+y)-\mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})\right)\Big|_0^1$$
$$+L_+(\frac{1-\alpha_2-\alpha_3}{\alpha_3},\frac{\alpha_4+1}{\alpha_4-\alpha_1},\frac{1}{\alpha_1-\alpha_4},-\frac{\alpha_2+\alpha_3}{\alpha_3})+$$
$$\int_0^1 \frac{-\ln(\alpha_3)\mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})+\ln(\alpha_1-\alpha_4)\ln(\alpha_4+y)\ln(1-\alpha_2-\alpha_3 y)}{\frac{\alpha_2}{\alpha_3}+y}dy;$$

$$\alpha_4 < -1, \alpha_3 > 0 \quad : \quad \mathrm{Li}_2(\alpha_2+\alpha_3 y)\left(\ln(\alpha_1-\alpha_4)\ln(-\alpha_4-y)-\mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})\right)\Big|_0^1$$
$$+L_+(\frac{1-\alpha_2-\alpha_3}{\alpha_3},\frac{\alpha_4+1}{\alpha_4-\alpha_1},\frac{1}{\alpha_1-\alpha_4},-\frac{\alpha_2+\alpha_3}{\alpha_3})$$
$$+\int_0^1 \frac{-\ln(\alpha_3)\mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})+\ln(\alpha_1-\alpha_4)\ln(-\alpha_4-y)\ln(1-\alpha_2-\alpha_3 y)}{\frac{\alpha_2}{\alpha_3}+y}dy;$$

$$\alpha_1 > \alpha_4 > 0, \alpha_3 < 0 \quad : \quad \mathrm{Li}_2(\alpha_2+\alpha_3 y)\left(\ln(\alpha_1-\alpha_4)\ln(\alpha_4+y)-\mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})\right)\Big|_0^1$$
$$-L_+(\frac{\alpha_2-1}{\alpha_3},\frac{\alpha_4}{\alpha_4-\alpha_1},\frac{1}{\alpha_4-\alpha_1},\frac{\alpha_2}{\alpha_3})+$$
$$\int_0^1 \frac{-\ln(-\alpha_3)\mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})+\ln(\alpha_1-\alpha_4)\ln(\alpha_4+y)\ln(1-\alpha_2-\alpha_3 y)}{\frac{\alpha_2}{\alpha_3}+y}dy;$$

$$\alpha_4 < -1, \alpha_3 < 0 \quad : \quad \mathrm{Li}_2(\alpha_2+\alpha_3 y)\left(\ln(\alpha_1-\alpha_4)\ln(-\alpha_4-y)-\mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})\right)\Big|_0^1$$
$$-L_+(\frac{\alpha_2-1}{\alpha_3},\frac{\alpha_4}{\alpha_4-\alpha_1},\frac{1}{\alpha_4-\alpha_1},\frac{\alpha_2}{\alpha_3})+$$
$$\int_0^1 \frac{-\ln(-\alpha_3)\mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})+\ln(\alpha_1-\alpha_4)\ln(-\alpha_4-y)\ln(1-\alpha_2-\alpha_3 y)}{\frac{\alpha_2}{\alpha_3}+y}dy$$

$$L_+(\alpha_1,\alpha_2,\alpha_3,\alpha_4) =$$

$$\begin{cases} \begin{array}{c}\alpha_3>0\\ 0\le\alpha_2\le 1,\ \alpha_2+\alpha_3\le 1\end{array} & : \quad \begin{array}{l} -L_+(\alpha_1,1-\alpha_2,-\alpha_2,\alpha_4) - L_{-++}(\frac{1-\alpha_2}{\alpha_3},\alpha_1,\frac{\alpha_2}{\alpha_3},\alpha_4)+ \\ \int\limits_0^1 \frac{\ln(\alpha_1+y)}{\alpha_4+y}\left(\zeta(2)-\ln(\alpha_3)[\ln(\alpha_2+\alpha_3 y)+\ln(\frac{1-\alpha_2}{\alpha_3}-y)]\right)dy; \end{array} \\ \begin{array}{c}\alpha_3<0\\ 0\le\alpha_2\le 1,\ 0\le\alpha_2+\alpha_3\end{array} & : \quad \begin{array}{l} -L_+(\alpha_1,1-\alpha_2,-\alpha_2,\alpha_4) - L_{-++}(-\frac{\alpha_2}{\alpha_3},\alpha_1,\frac{\alpha_2-1}{\alpha_3},\alpha_4)+ \\ \int\limits_0^1 \frac{\ln(\alpha_1+y)}{\alpha_4+y}\left(\zeta(2)-\ln(-\alpha_3)[\ln(\alpha_2+\alpha_3 y)+\ln(\frac{\alpha_2-1}{\alpha_3}+y)]\right)dy. \end{array} \end{cases}$$

Multiple polylogarithms (Goncharov (1998))

$$Li_{m_k,\ldots,m_1}(x_k,\ldots,x_1) = \lim_{n_1\to\infty} \sum_{n_1>n_2\ldots>n_k>0} \frac{x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}}{n_1^{m_1} n_2^{m_2} \ldots n_k^{m_k}}, \ |x_i| < 1$$

$$Li_{m_k,\ldots,m_1}(x_k,\ldots,x_1) = \int_0^{x_1 x_2 \ldots x_k} \left(\frac{dt}{t}\circ\right)^{m_1-1} \frac{dt}{x_2 x_3 \ldots x_k - t} \circ$$
$$\left(\frac{dt}{t}\circ\right)^{m_2-1} \frac{dt}{x_3 \ldots x_k - t} \circ \ldots \circ \left(\frac{dt}{t}\circ\right)^{m_k-1} \frac{dt}{1-t},$$

where the iterated integrals are defined by:

$$\int_0^{\lambda} \frac{dt}{a_n - t} \circ \ldots \circ \frac{dt}{a_1 - t} = \int_0^{\lambda} \frac{dt_n}{a_n - t_n} \int_0^{t_n} \frac{dt_{n-1}}{a_{n-1} - t_{n-1}} \times \ldots \times \int_0^{t_2} \frac{dt_1}{a_1 - t_1}.$$

k - depth, $w = \sum_{i=1}^{k} m_i$ - weight

$Nielsen's\ polylogarithms:$

$$S_{n,p}(x) = \frac{(-1)^{n-1+p}}{(n-1)!p!} \int_0^1 dt \frac{\ln^{n-1}(t)\ln^p(1-tx)}{t} \quad \Rightarrow \quad S_{n,p}(x) = Li_{1,\dots,1,n+1}(\underbrace{1,\dots,1}_{p-1},x)$$

$Harmonic\ polylogarithms:$

$$H_0(x) = \ln(x),\, H_1(x) = -\ln(1-x),$$
$$H_{-1}(x) = \ln(1+x),$$
$$H_{m_1+1,m_2,\dots,m_k}(x) = \int_0^x dt f_0(t) H_{m_1,m_2,\dots,m_k}(t), \quad \Rightarrow \quad H_{m_1,\dots,m_k}(x) = Li_{m_k,\dots,m_1}(\underbrace{1,\dots,1}_{k-1},x)$$
$$H_{\pm 1,m_2,\dots,m_k}(x) = \int_0^x dt f_{\pm 1}(t) H_{m_2,\dots,m_k}(t),$$
$$f_0(x) = \tfrac{1}{x},\, f_1(x) = \tfrac{1}{1-x},\, f_{-1}(x) = \tfrac{1}{1+x}.$$

$Two\ dimensional\ Harmonic\ polylogarithms:$

$$f(z,x) = \tfrac{1}{z+x},\ f(1-z,x) = \tfrac{1}{1-z-x} \qquad \Downarrow$$

$$\int_0^x dt f(z,t) \mathrm{Li}_{m_k,\dots,m_1}\left(x_k,\dots,x_2,\frac{t}{x_2\dots x_k}\right) = -\mathrm{Li}_{m_k,\dots,m_1,1}\left(x_k,\dots,x_2,\frac{-z}{x_2\dots x_k},\frac{-x}{z}\right)$$

$$\int_0^x dt f(1-z,t) \mathrm{Li}_{m_k,\dots,m_1}\left(x_k,\dots,x_2,\frac{t}{x_2\dots x_k}\right) = \mathrm{Li}_{m_k,\dots,m_1,1}\left(x_k,\dots,x_2,\frac{1-z}{x_2\dots x_k},\frac{x}{1-z}\right)$$

▲ L-functions in terms of multiple polylogarithms

Domain for $L_{\pm++}$

$$L_{+++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \quad : \quad \alpha_1>0, \alpha_2>0, \alpha_3>0, \alpha_4<-1 \quad \text{or} \quad \alpha_4>0;$$
$$L_{-++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \quad : \quad \alpha_1>1, \alpha_2>0, \alpha_3>0, \alpha_4<-1 \quad \text{or} \quad \alpha_4>0;$$

$$\begin{aligned}
\Rightarrow L_{\pm++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= Li_{1,1,1,1}\left(\pm\tfrac{\alpha_4}{\alpha_1},\tfrac{\alpha_2}{\alpha_4},\tfrac{\alpha_3}{\alpha_2},-\tfrac{1}{\alpha_3}\right) \\
+Li_{1,1,1,1}\left(\pm\tfrac{\alpha_4}{\alpha_1},\tfrac{\alpha_3}{\alpha_4},\tfrac{\alpha_2}{\alpha_3},-\tfrac{1}{\alpha_2}\right) &+ \ln\alpha_1 Li_{1,1,1}\left(\tfrac{\alpha_4}{\alpha_2},\tfrac{\alpha_3}{\alpha_4},-\tfrac{1}{\alpha_3}\right) \\
+\ln(1+\alpha_2)Li_{1,1,1}\left(\pm\tfrac{\alpha_4}{\alpha_1},\tfrac{\alpha_3}{\alpha_4},-\tfrac{1}{\alpha_3}\right) &+ \ln(1+\alpha_3)Li_{1,1,1}\left(\pm\tfrac{\alpha_4}{\alpha_1},\tfrac{\alpha_2}{\alpha_4},-\tfrac{1}{\alpha_2}\right) \\
+\ln\alpha_1\ln\alpha_2 Li_{1,1}\left(\tfrac{\alpha_4}{\alpha_3},-\tfrac{1}{\alpha_4}\right) &+ \ln\alpha_1\ln(1+\alpha_3)Li_{1,1}\left(\tfrac{\alpha_4}{\alpha_2},-\tfrac{1}{\alpha_4}\right) \\
+\ln(1+\alpha_2)\ln(1+\alpha_3)Li_{1,1}\left(\pm\tfrac{\alpha_4}{\alpha_1},-\tfrac{1}{\alpha_4}\right) &+ \ln\alpha_1\ln\alpha_2\ln\alpha_3\ln\left(\tfrac{\alpha_4+1}{\alpha_4}\right)
\end{aligned}$$

Boundary values $\{\pm1,0\}$ can still be used in $L_{\pm++}$. We consider these cases separately.

Domain for $L_+(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$

$$\alpha_1 > 0, \alpha_2 \le 1, \alpha_2 + \alpha_3 \le 1, \alpha_3 \neq 0, \alpha_4 < -1 \quad \text{or} \quad \alpha_4 > 0.$$

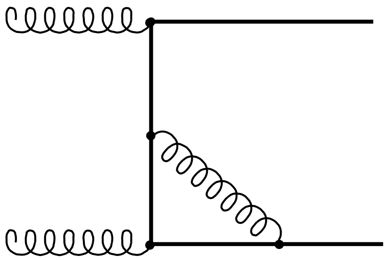
$$\begin{aligned}
\Rightarrow L_+(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \quad &= \quad Li_{2,1,1}\left(\alpha_2 - \alpha_3\alpha_4, \tfrac{\alpha_2 - \alpha_1\alpha_3}{\alpha_2 - \alpha_3\alpha_4}, \tfrac{\alpha_2}{\alpha_2 - \alpha_1\alpha_3}\right) \\
-Li_{2,1,1}\left(\alpha_2 - \alpha_3\alpha_4, \tfrac{\alpha_2 - \alpha_1\alpha_3}{\alpha_2 - \alpha_3\alpha_4}, \tfrac{\alpha_2 + \alpha_3}{\alpha_2 - \alpha_1\alpha_3}\right) \quad &+ \quad \ln\alpha_1 Li_{2,1}\left(\alpha_2 - \alpha_3\alpha_4, \tfrac{\alpha_2}{\alpha_2 - \alpha_3\alpha_4}\right) \\
-\ln(\alpha_1 + 1) Li_{2,1}\left(\alpha_2 - \alpha_3\alpha_4, \tfrac{\alpha_2 + \alpha_3}{\alpha_2 - \alpha_3\alpha_4}\right), \quad & \quad \alpha_2 - \alpha_3\alpha_4 \neq 0, \alpha_2 - \alpha_1\alpha_3 \neq 0
\end{aligned}$$

Boundary values $\{-1, 0\}$ can still be used in L_+. We consider these cases separately as well as cases $\alpha_2 - \alpha_3\alpha_4 = 0$ and $\alpha_2 - \alpha_1\alpha_2 = 0$.

- maximal weight and depth of multiple polylogarithms is 4
- all relations have been checked numerically using the program of J.Vollinga and S.Weinzierl for numerical evaluation of multiple polylogarithms hep-ph/0410259

Laurent series expansion of one–loop amplitudes including spin

$$M = \varepsilon^b_\mu(p_1)\varepsilon^a_\nu(p_2)\bar{u}(p_3)M^{\mu\nu,ab}u(p_4).$$



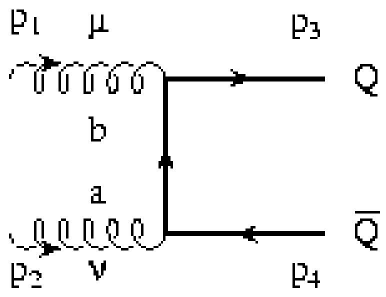
3c2

$$2 - \text{point function :} \qquad B_j = iC_\varepsilon(m^2)\left(\sum_{n=-1}^{N} \varepsilon^n B_j^{(n)}\right)$$

$$3 - \text{point function :} \qquad C_j = iC_\varepsilon(m^2)\left(\sum_{n=-2}^{N} \varepsilon^n C_j^{(n)}\right)$$

For the graph (3c2) we obtain:

$$
\begin{aligned}
M_{(3c2)}^{\mu\nu,ab} &= B_t^{\mu\nu,ab}\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}(6m^2/t+1)+2B_1^{(k-1)}z_t/t+2C_2^{(k)}m^2+4C_2^{(k-1)}m^2-k8(4m^2/t+1)\}/6 \\
&+ iT^bT^a\Big(p_4^\nu\gamma^\mu\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}(-2m^2/t-3+t/T)-B_1^{(k-1)}t/T-2C_2^{(k)}m^2+k8T/t\} \\
&+ mp_4^\nu(2p_3^\mu-\gamma^\mu\not{p}_1)\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}/T-B_1^{(k-1)}(2/t+1/T)-2C_2^{(k-1)}+k4/t\} \\
&- m\gamma^\mu\gamma^\nu\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}+B_1^{(k-1)}+C_2^{(k-1)}t-6k\}\Big)/(3t)\,.
\end{aligned}
$$

$B_t^{\mu\nu,ab}=iT^bT^a(\gamma^\mu\not{p}_1\gamma^\nu-2p_3^\mu\gamma^\nu)/t$ - Born term for 

Concluding remarks

- Scalar one-loop results up to $\mathcal{O}(\varepsilon^2)$ in terms of L-functions are available at Phys. Rev. D **71**, 054028 (2005). In format of MATHEMATICA: EPAP No. E-PRVDAQ-71-025507
- Scalar one-loop results up to $\mathcal{O}(\varepsilon^2)$ in terms of multiple polylogarithms will be available soon
- Compare efficiency of the results in terms of L-functions with results in terms of multiple polylogarithms
- quark initiated amplitudes are finished
  gluon initiated amplitudes - work in progress
- squaring of the loop amplitudes in the future