The Abelian sandpile model: towards a lattice realization of a logarithmic CFT

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Forword

Historically, sandpile models have been proposed by Bak, Tang & Wiesenfeld ('87) as prototypes of self-organized critical models (SOC).

Idea was: many critical behaviours (power laws) in nature, but unlikely to result from fine-tuning $\longrightarrow$ it is the dynamics that drives the system to a critical state, even if the system is prepared in a non-critical state.

Example (BTW) = sandpile, with slow addition of sand (pile builds up, then avalanches of all sizes).

[Deepak Dhar, Theoretical studies of self-organized criticality, Physica A 369 (2006) 29-70]

Important for us:

1. interesting non-equilibrium system, with stationary measure
2. lattice realization of logarithmic CFT (light on subtleties)

Plan

1. The Abelian sandpile model (following Dhar)
   definition of 2+1 – invariant measure – Abelian property – recurrent configurations – spanning trees – $c = -2$ – boundary conditions
2. Logarithmic CFT
   non-diagonalizable L_0 – Jordan blocks – typical example of $c = -2$
3. Lattice observables in ASM $\leftrightarrow$ LCFT
   dissipation – change of boundary conditions – height variables
4. Conclusions

– Part I –
The Abelian sandpile model

The model

Take a grid Λ with N sites

Attach a random variable $h_i = 1, 2, 3, 4$ to every site (h_i is # grains)

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	1	1	3	4	3	4
4	4	3	2	4	3	2	1	2	3
2	3	3	4	4	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	2	4	3	1	2	3	4	1

stable configs $= 4^N$

Dynamics

The sandpile model is a stochastic dynamical system in discrete $2+1$.

Dynamics takes $\mathcal{C}_t$ into $\mathcal{C}_{t+1}$ in two steps:

1. on random site i, drop one grain: $h_i \to h_i + 1$
2. relaxation: all unstable sites topple (avalanche)

   If $h_i \geq 5$, then $\begin{cases} h_i \to h_i - 4 \\ h_j \to h_j + 1, \quad j = \text{nearest neighbour of } i \end{cases}$

   Until all sites are stable again $\longleftarrow$ OK BECAUSE DISSIPATION !!
   Resulting configuration is $\mathcal{C}_{t+1}$.

Potential chain reaction: one grain dropped can trigger a large avalanche.
System spanning avalanches will happen, and induce correlations of heights over long distances $\longrightarrow$ critical state

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	1	1	3	4	3	4
4	4	3	2	4	3	2	1	2	3
2	3	3	4	4	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	1	1	3	4	3	4
4	4	3	2	5	3	2	1	2	3
2	3	3	4	4	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

$\longrightarrow$

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**3**	**1**	**4**	2	1	2	3
2	3	3	4	5	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**3**	**2**	**4**	2	1	2	3
2	3	3	5	**1**	**4**	1	1	2	3
2	3	2	4	**4**	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**3**	**1**	**4**	2	1	2	3
2	3	3	4	5	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

$\longleftarrow$

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**3**	**2**	**4**	2	1	2	3
2	3	3	5	**1**	**4**	1	1	2	3
2	3	2	4	**4**	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

$\longrightarrow$

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**1**	**2**	**4**	1	1	2	3
2	3	2	5	**4**	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**2**	**4**	1	1	2	3
2	3	**3**	**1**	5	3	4	2	4	3
3	1	3	**3**	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

$\longleftarrow$

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**1**	**2**	**4**	1	1	2	3
2	3	2	5	**4**	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**2**	**4**	1	1	2	3
2	3	**3**	**1**	5	3	4	2	4	3
3	1	3	**3**	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

$\longrightarrow$

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**1**	**4**	4	2	4	3
3	1	3	**3**	5	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**2**	**4**	4	2	4	3
3	1	3	**4**	**1**	**3**	1	4	4	3
4	3	4	4	5	1	2	3	4	1

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**1**	**4**	4	2	4	3
3	1	3	**3**	5	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

$\longleftarrow$

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**2**	**4**	4	2	4	3
3	1	3	**4**	**1**	**3**	1	4	4	3
4	3	4	4	5	1	2	3	4	1

$\longrightarrow$

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**2**	**4**	4	2	4	3
3	1	3	**4**	**2**	**3**	1	4	4	3
4	3	4	5	**1**	**2**	2	3	4	1

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**2**	**4**	4	2	4	3
3	1	3	5	**2**	**3**	1	4	4	3
4	3	5	**1**	**2**	**2**	2	3	4	1

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**2**	**4**	4	2	4	3
3	1	3	**4**	**2**	**3**	1	4	4	3
4	3	4	5	**1**	**2**	2	3	4	1

$\longleftarrow$

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**2**	**2**	**4**	4	2	4	3
3	1	3	5	**2**	**3**	1	4	4	3
4	3	5	**1**	**2**	**2**	2	3	4	1

$\longrightarrow$

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**3**	**2**	**4**	4	2	4	3
3	1	5	**1**	**3**	**3**	1	4	4	3
4	**4**	**1**	**3**	**2**	**2**	2	3	4	1

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**4**	**3**	**2**	**4**	4	2	4	3
3	**2**	1	**2**	**3**	**3**	1	4	4	3
4	**4**	**2**	**3**	**2**	**2**	2	3	4	1

$\longleftarrow$

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**3**	**3**	**2**	**4**	4	2	4	3
3	1	5	**1**	**3**	**3**	1	4	4	3
4	**4**	**1**	**3**	**2**	**2**	2	3	4	1

Typical avalanche

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	1	1	3	4	3	4
4	4	3	2	4	3	2	1	2	3
2	3	3	4	4	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	4	4	4	1	2	3	4	1

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	**2**	1	3	4	3	4
4	4	3	**4**	**2**	**4**	2	1	2	3
2	3	**4**	**2**	**3**	**4**	1	1	2	3
2	3	**4**	**3**	**2**	**4**	4	2	4	3
3	**2**	**1**	**2**	**3**	**3**	1	4	4	3
4	**4**	**2**	**3**	**2**	**2**	2	3	4	1

11 topplings, 22 sites affected, 3 grains fell off, into the sink.

The order of topplings does not matter.

Seeding operators

Seeding operators a_i: act on stable configurations by dropping one grain on site i and by letting the configuration relax.

Sandpile dynamics = each unit of time, a_i is applied with (uniform) probability $p_i = \frac{1}{N}$.

Because order of topplings does not matter, one can show

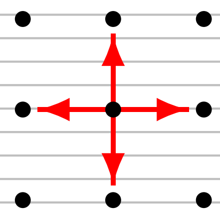
$$[a_i, a_j] = 0 \qquad \forall i, j$$

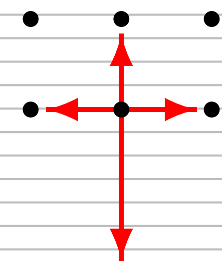
(Essentially, because toppling condition is ultra-local.)

They form an Abelian algebra, soon to be promoted to an Abelian group.

Laplacian

Redistribution of sand to neighbour sites:

bulk : 

boundary : 

If $h_i \geq 5$, then $\begin{cases} h_i \to h_i - 4 \\ h_{\text{n.n.}} \to h_{\text{n.n.}} + 1 \end{cases} \iff h_j \to h_j - \Delta_{ij} \quad \forall j$

Toppling matrix Δ is simply the Laplacian with <u>open</u> (Dirichlet) boundary conditions,

$$\Delta_{ij} = \begin{cases} 4 & \text{for } i = j \\ -1 & \text{for } \langle i, j \rangle \end{cases}$$

Bulk sites are conservative, open boundary sites are dissipative: when i topples, $\sum_j \Delta_{ij}$ grains leave the system, or “transferred to the sink”.

Master equation

Dynamics is stochastic because seeding of sand is random.

If $P_t(\mathcal{C})$ is probability distribution at time t, then (Markov chain)

$$P_{t+1}(\mathcal{C}) = \sum_{i\in\Lambda} p_i \sum_{\mathcal{C}'} \delta(\mathcal{C} - a_i\mathcal{C}')\, P_t(\mathcal{C}')$$

The a_i are not invertible on the stable configurations: $\mathcal{C}_{\rm min} = \{h_i = 1\}_i$ is not in the image of the seeding operators $\Longrightarrow P_t(\mathcal{C}_{\rm min}) = 0$.

This is general. Configurations are either

- transient: they are not in the repeated image of the dynamics, and occur only a finite number of times $\Rightarrow P_t(\mathcal{C}) = 0$ for large enough t
- recurrent: they are in the repeated image of the dynamics and asymptotically occur with non-zero probability; $\exists m_i \;:\; a_i^{m_i}\mathcal{C} = \mathcal{C}$.

Invariant measure

→ time evolution flow towards recurrent configurations

→ set $\mathcal{R}$ of recurrent configurations is closed under the dynamics

→ seeding operators a_i are invertible on $\mathcal{R}$ → generate Abelian group

Behaviour of sandpile controlled by invariant measure(s) $\lim_{t\to\infty} P_t$.

We have the first important result:

The invariant measure P^*_Λ is unique and is uniform on the recurrent set $\mathcal{R}$

$$P^*_\Lambda(\mathcal{C}) = \begin{cases} \frac{1}{|\mathcal{R}|} & \text{if } \mathcal{C} \text{ is recurrent} \\ 0 & \text{if } \mathcal{C} \text{ is transient} \end{cases}$$

P^*_Λ depends on type of lattice, size of lattice, boundary conditions, number of dissipative sites, dissipation rates, ...

Recurrent set

Number of recurrent configurations ?
The group G generated by the a_i's acts irreducibly on $\mathcal{R}$: any $\mathcal{C}$ is obtained from any $\mathcal{C}'$ by a g, equivalently $\mathcal{R} = G\,\mathcal{C}^*$, for a fixed $\mathcal{C}^*$.
Therefore $|\mathcal{R}|$ is the order of G.

G is not freely generated by the a_i's, because $\prod_j a_j^{\Delta_{ij}} = 1, \forall i$.

Since G is finite Abelian, we can represent $a_j = \mathrm{e}^{2\mathrm{i}\pi\phi_j}$, such that $\sum_j \Delta_{ij}\,\phi_j = m_i$ are integers $\quad\Longrightarrow\quad \phi_j = \sum_i \Delta^{-1}_{jk}\,m_k$.

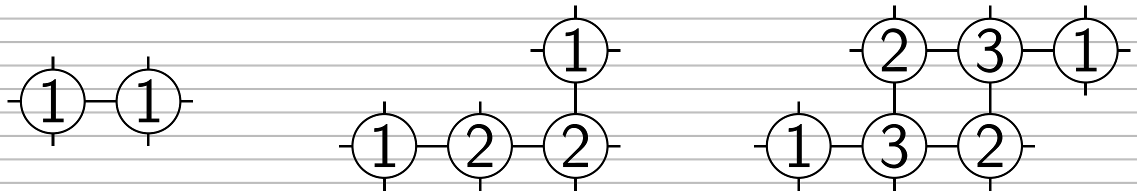
However $\{m_k\}$ and $\{m_k + \sum_l \Delta_{kl} n_l\}$ yield identical phases.

Thus distinct representations of G are labelled by integer vectors $\{m_k\}$ modulo the lattice generated by the columns $\{\Delta_{kl}\}_l$:

$$\boxed{|\mathcal{R}| = |G| = \det \Delta} \qquad (\sim 3.21^N \ll 4^N)$$

Characterization

The minimal configuration $\mathcal{C}_{\min} = \{h_i = 1\}$ is clearly not recurrent. Likewise, configurations containing the following clusters cannot be recurrent:



Forbidden Sub–Configuration: cluster F of sites s.t. every i in F has height $h_i \leq$ number of nearest neighbours in F.

A configuration is recurrent iff it has no FSCs

- Non-local characterization: requires to scan the whole configuration, and induces long range correlations of the height variables
- Makes the sandpile model a complex system: difficult to separate different length scales.

Burning algorithm

To make sure a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

4	3	1	2
2	3	2	3
1	3	2	4
2	3	4	2

Burning algorithm

To make sure a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

	3	1	2
2	3	2	3
1	3	2	
2	3		2

Burning algorithm

To make sure a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

	3	1	2
2	3	2	3
1	3	2	
2	3		2

Burning algorithm

To make sure a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

		1	2
2	3	2	
1	3	2	
2			

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		1
2	3	2
1	3	2

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$$\text{the configuration } \boxed{\begin{matrix} 4 & 3 & \color{red}{1} & 2 \\ \color{red}{2} & \color{red}{3} & \color{red}{2} & 3 \\ \color{red}{1} & \color{red}{3} & \color{red}{2} & 4 \\ 2 & 3 & 4 & 2 \end{matrix}} \text{ is not recurrent !}$$

Burning algorithm

To make sure a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

the configuration

4	3	1	2
2	3	2	3
1	3	2	4
2	3	4	2

is not recurrent !

but the configuration

4	3	1	2
3	3	2	3
1	3	2	4
2	3	4	2

is recurrent !

Burning algorithm

To make sure a configuration contains no FSC, we apply the **burning algorithm**: we successively burn all sites with heights strictly larger than the number of unburnt neighbours; the sites which cannot be burnt form an FSC.

but the configuration

4	3	1	2
3	3	2	3
1	3	2	4
2	3	4	2

is recurrent !

The burning algorithm does more: keeping track of the way fire spreads in the lattice leads to spanning trees ...

Spanning trees

That a site is burnable at a certain instant implies that at least one of its neighbours was burnt the instant before: at initial time, fire is located in the sink and ignites boundary sites $\longrightarrow$ the fire propagates from neighbours to neighbours.

This fire line defines a spanning tree.

4	3	1	2
3	3	2	3
1	3	2	4
2	3	4	2

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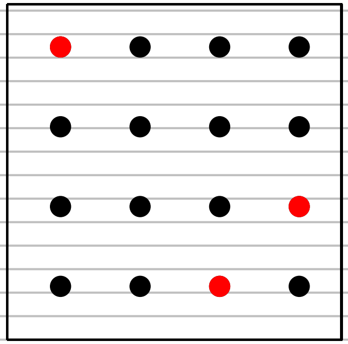
3	3	2	3
1	3	2	4
2	3	4	2



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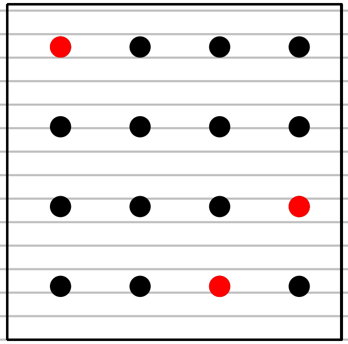

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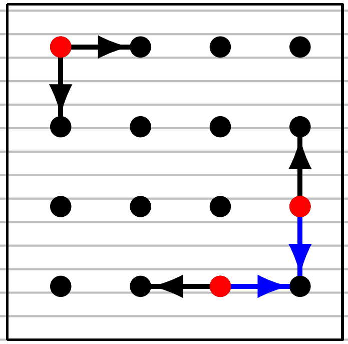

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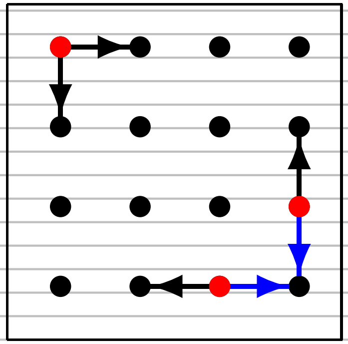

Use a prescription to select a blue arrow:

$$2 \text{ (height)} - 0 \text{ (\# unburnt neigh.)} = 2 \longrightarrow \text{second in } \{N,E,S,W\}$$

That a site is burnable at a certain instant implies that at least one of its neighbours was burnt the instant before: at initial time, fire is located in the sink and ignites boundary sites $\longrightarrow$ the fire propagates from neighbours to neighbours.

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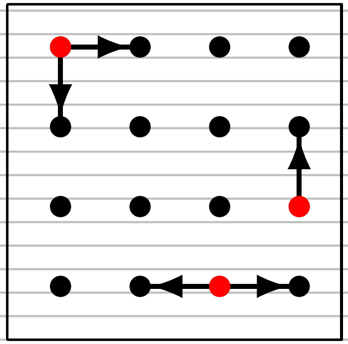

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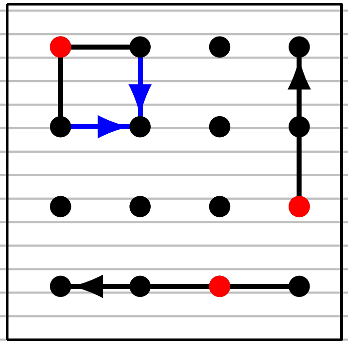
That a site is burnable at a certain instant implies that at least one of its neighbours was burnt the instant before: at initial time, fire is located in the sink and ignites boundary sites $\longrightarrow$ the fire propagates from neighbours to neighbours.

This fire line defines a spanning tree.

	3	1	2
3	3	2	3
1	3	2	
2	3		2



Use same prescription to select a blue arrow:

3 (height) $-$ 2 ($\#$ unburnt neigh.) $= 1 \longrightarrow$ first in {N,E,S,W}

Spanning trees

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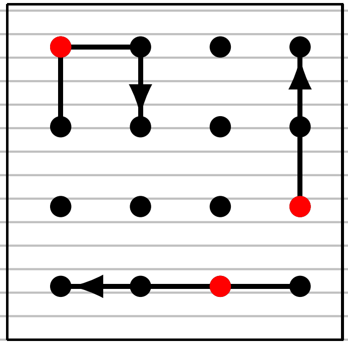
	3	1	2
3	3	2	3
1	3	2	
2	3		2



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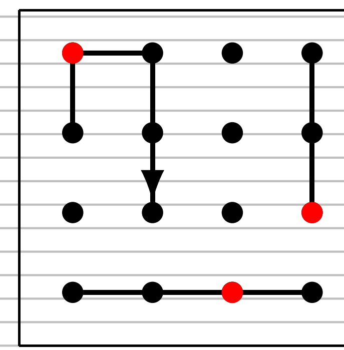
That a site is burnable at a certain instant implies that at least one of its neighbours was burnt the instant before: at initial time, fire is located in the sink and ignites boundary sites $\longrightarrow$ the fire propagates from neighbours to neighbours.

This fire line defines a spanning tree.

		1	2
	3	2	
1	3	2	
2			



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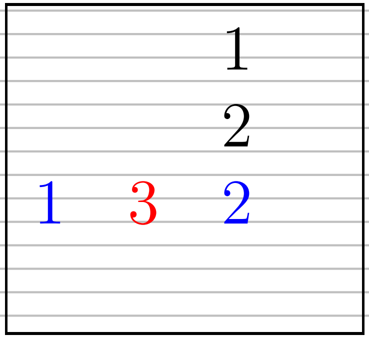

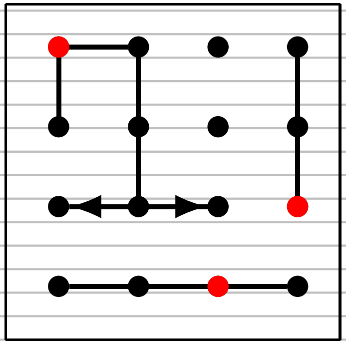

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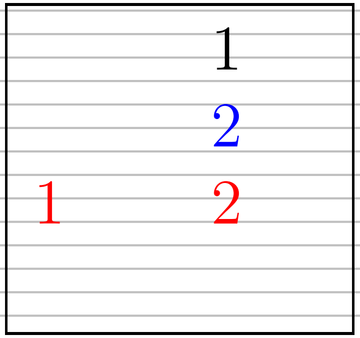

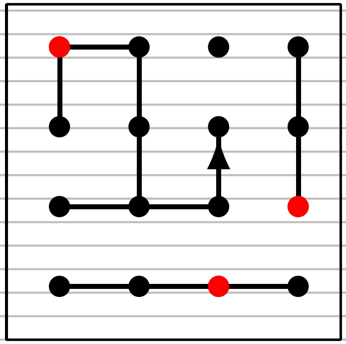

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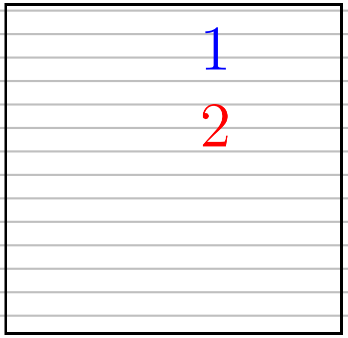
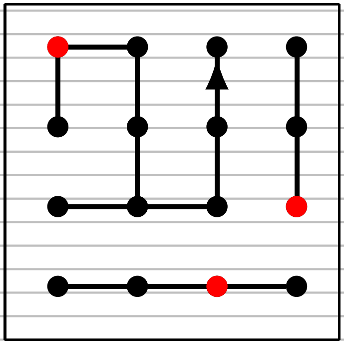
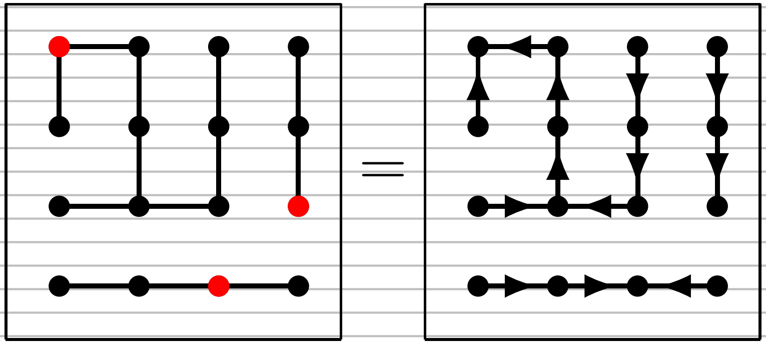




Spanning trees

This fire line defines a (disconnected) spanning tree.

4	3	1	2
3	3	2	3
1	3	2	4
2	3	4	2



Spanning tree grows from roots (red dots), which are always dissipative sites (connected to the sink).

With the prescription used, we have

$$\text{recurrent configurations} \stackrel{1:1}{\longleftrightarrow} \text{spanning trees}$$

(Kirchhoff's theorem)

ASM: so far

1. defined on a finite grid Λ, with heights $h_i = 1, 2, 3, 4$
2. necessity of dissipation (sites connected to sink)
3. configurations are either recurrent or transient
4. recurrent are in 1-to-1 correspondence with spanning trees growing from dissipative sites
5. dynamics has a unique invariant measure P^*_Λ, uniform on recurrent configurations or on spanning trees
6. non-local:
   heights are local microscopic variables but globally constrained

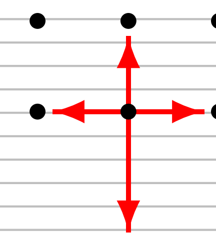
   $$\updownarrow$$

   spanning trees are unconstrained but global variables

Boundary conditions

- open boundary site (dissipative)
  Under toppling, loses 4, gives 1 to three neighbours

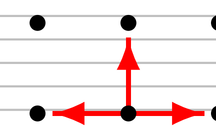
$$\Delta_{ii} = 4, \quad \Delta_{\langle ij \rangle} = -1, \qquad \sum_{j \in \Lambda} \Delta_{ij} > 0$$



Height variable $1 \leq h_{\rm open} \leq 4$.

- closed boundary site (conservative)
  Under toppling, loses 3, gives 1 to three neighbours

$$\Delta_{ii} = 3, \quad \Delta_{\langle ij \rangle} = -1, \qquad \sum_{j \in \Lambda} \Delta_{ij} = 0$$



Height variable $1 \leq h_{\rm closed} \leq 3$.

Note: all sites closed implies $\sum_j \Delta_{ij} = 0 \; \forall i \Rightarrow \det \Delta = |\mathcal{R}| = 0$.

B.c. (cont'd)

- boundary arrows (in spanning tree variables)
  Trees are constrained to contain certain boundary bonds, with an arrow indicating the direction to the root

- periodic boundary condition
  Cylindrical geometry can be imposed provided there remain dissipation on the boundaries (torus not allowed)

- others ???

ASM: summary

1. defined on a finite grid Λ, with heights $h_i = 1, 2, 3, 4$ with prescribed boundary conditions (open, closed, arrows, ...) $\longrightarrow$ specific Δ
2. **necessity of dissipation** (sites connected to sink)
3. configurations are either recurrent or transient
4. recurrent are in 1-to-1 correspondence with spanning trees growing from dissipative sites
5. dynamics has a unique invariant measure P^*_Λ, uniform on recurrent configurations or on spanning trees
6. non-local:
   heights are local microscopic variables but globally constrained

   $$\updownarrow$$

   spanning trees are unconstrained but global variables

Want to show

The thermodynamic limit $\lim_{|\Lambda|\to\infty} P^*_\Lambda$ of the invariant measure is a quantum field theoretic measure of a (logarithmic) conformal field theory

First hint at $c = -2$

Partition function measures the effective degrees of freedom

$$Z_\Lambda = |\mathcal{R}| = \det \Delta$$

Finite-size correction: rectangle $L \times M$ with open b.c.

$$\lim_{M\to\infty} \frac{1}{M} \log Z_\Lambda = \frac{4G}{\pi} L + \left(\frac{4G}{\pi} - \log(1+\sqrt{2}) \right) - \frac{\pi}{12L} + \cdots$$

First term is bulk entropy per site: $f_{\text{bulk}} = \exp \frac{4G}{\pi} \simeq 3.21$

Second term: $f_{\text{open}} = \exp[\frac{6G}{\pi} - \frac{1}{2}\log(1+\sqrt{2})] \simeq 3.70$

Blue term identified with $\dfrac{\pi c}{24L}$ $\implies$ $c = -2$

Questions

To confirm the relevance of conformal description, ask questions that have an answer in CFT:

1. Correlations of height variables
2. Effect of changing the boundary conditions
3. Effect of introducing additional dissipation

– Part II –
Logarithmic conformal field theory
(at $c = -2$)

Rational models

Usual features of rational models:

1. finite number of Virasoro representations
2. Vir representations are highest weight, completely reducible
3. Vir representations mainly identified by a conformal weight (L_0 diagonalizable)
4. conformal weights are bounded below
5. full, non-chiral theory basically reduces to chiral parts
6. correlation functions only have algebraic singularities
7. finite fusion (or quasi-rational)
8. chiral characters transform linearly under modular group of torus

Log CFTs

Typical features of logarithmic models:

1. finite number of Virasoro representations YES/NO
2. Vir representations are highest weight, completely reducible NO
3. Vir representations mainly identified by a conformal weight NO (L_0 diagonalizable)
4. conformal weights are bounded below YES
5. full, non-chiral theory basically reduces to chiral parts NO
6. correlation functions only have algebraic singularities NO, Log^k
7. finite fusion (or quasi-rational) YES
8. chiral characters transform linearly under modular group NO

Minimal models

Minimal models are parametrized by (p, p'):

$$c = 1 - \frac{6(p-p')^2}{pp'}$$

Kac table of conformal weights

$$h_{r,s} = \frac{(p'r - ps)^2 - (p-p')^2}{4pp'}, \qquad \text{(usually truncated)}$$

non-empty for $p, p' \geq 2$.

However the value of the central charge relevant here is

$$c = -2 \quad \longleftrightarrow \quad p = 1,\, p' = 2$$

Full Kac table

We take KT as a guiding principle : $h_{1,s} = \frac{(s-2)^2-1}{8}$, $\quad s = 1, 2, 3, \cdots$

$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$
1	0	0	1	3	6	10
$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$
0	0	1	3	6	10	15
$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\frac{143}{8}$
0	1	3	6	10	15	21

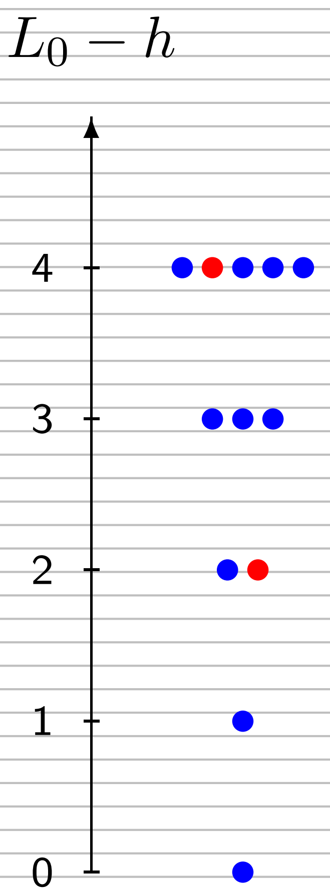
We observe: $-\frac{1}{8}$ is smallest, the only negative

Δh is an integer for many pairs. Required for LogCFT !

Highest weight reps

Built on highest weight state $|h\rangle = \phi_h|0\rangle$ satisfying

$$L_0|h\rangle = h|h\rangle, \qquad L_p|h\rangle = 0 \quad \forall p > 0.$$



Verma module $\mathcal{M}_h$ is freely spanned by the action of the negative modes on $|h\rangle$

$$L_0(L_{-p_1} \cdots |h\rangle) = (h + p_1 + \cdots)(L_{-p_1} \cdots |h\rangle).$$

At finite level $N = L_0 - h$, finite number $p(N)$ of states, some of them singular (h.w.), i.e. satisfying

$$L_0|s\rangle = (h+N)|s\rangle, \qquad L_p|s\rangle = 0 \quad \forall p > 0.$$

Singular states generate submodules:

$\longrightarrow$ allows quotients : Vir representations $\sim \mathcal{M}_h/$●

Reducible vs irreducible

Precise nature of quotients can be tricky : need to know whether higher level singular states are descendants of lower level singular states ... Complete answer by Feigin & Fuchs.

Situation simple for $c = -2$: all singular states are descendants of the lowest one ; all modules $\mathcal{M}_{r,s}$ have one singular state at level $N = rs$; corresponding quotient $\mathcal{V}_{r,s}$ is irreducible for yellow cells only.

$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$
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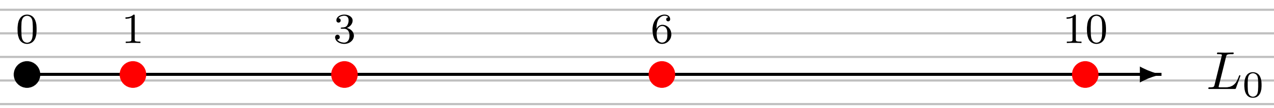
Examples

Verma module $\mathcal{M}_0$ for weight 0



$h_{1,1} = 0$

Irreducible quotient $\mathcal{V}_{1,1} = \mathcal{M}_0 / L_{-1}|h\rangle$ by first singular state.

Corresponding primary field satisfies $L_{-1}\phi_0(z) = \partial_z \phi_0(z) = 0$

$\longrightarrow$ ϕ_0 is the identity field $\mathbb{I}$.

$h_{1,3} = 0$

Reducible quotient $\mathcal{V}_{1,3} = \mathcal{M}_0 / (L_{-1}^2 - L_{-2}) L_{-1}|h\rangle$ by second singular state.

Corresponding primary field has zero weight, and is non-trivial (see later).

Fusion/OPE

Unlike in rational minimal models, h.w. $\mathcal{V}_{r,s}$ do not close under fusion !

Call μ the irreducible primary field of weight $h_{1,2} = -\frac{1}{8}$.

The singular field $[2L_{-1}^2 - L_{-2}]\mu = 0$ is null in quotient $\mathcal{V}_{1,2}$ and implies

$$\langle \mu(1)\mu(2)\mu(3)\mu(4)\rangle = (z_{12}z_{34})^{1/4}(1-x)^{1/4}\left[\alpha K(x) + \beta K(1-x)\right]$$

where $K(x) = \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1-x\sin^2 t}}$ has a log singularity at $x = 1$ ...

The log is unavoidable, either at $x = 0$ $(z_{12} = 0)$ or at $x = 1$ $(z_{23} = 1)$.

OPE reads

$$\mu(z)\mu(0) = \alpha z^{1/4}\left[\mathbb{I} + \cdots\right] + \beta z^{1/4}\left[\omega(0) + \mathbb{I}\log z + \cdots\right]$$

Jordan block

Second channel contains 2 fields, of weight 0

$$\mu(z)\mu(0) = z^{1/4}\left[\omega(0) + \mathbb{I}\,\log z + \cdots\right]$$

Peculiar under dilations $z \longrightarrow w = \lambda z$,

$$\mu(w)\mu(0) = w^{1/4}\left[\omega(0) - \mathbb{I}\,\log\lambda + \mathbb{I}\,\log z + \cdots\right],$$

the field ω picks inhomogeneous piece proportional to $\mathbb{I}$!

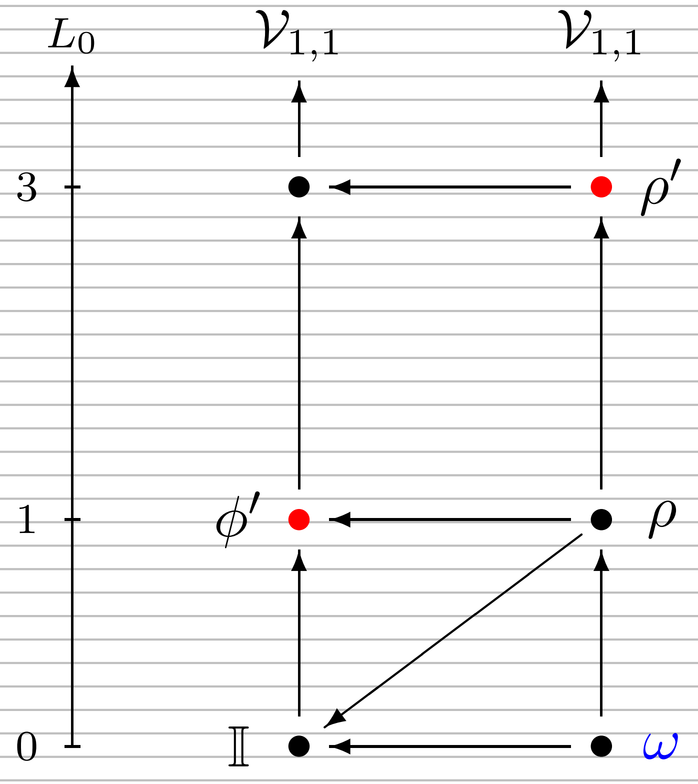
Particular case of general transformation of ω

$$\omega(w) = \omega(z) - \mathbb{I}\,\log\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right).$$

Implies

$$L_0\,\mathbb{I} = 0, \quad L_0\,\omega = \mathbb{I} \quad \longleftrightarrow \quad L_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Indecomposable representation



Defining relations of $\mathcal{R}_{1,1}$ are:

$$L_0\,\omega = \mathbb{I}, \qquad L_0\,\mathbb{I} = 0,$$

$$L_p\,\mathbb{I} = L_p\,\omega = 0, \qquad \forall p > 0$$

$$\phi' = L_{-1}\,\mathbb{I} \equiv 0,$$

$$\rho' = [L_{-1}^2 - L_{-2}]L_{-1}\,\omega \equiv 0.$$

Left $\mathcal{V}_{1,1}$ is a h.w. subrepres. of $\mathcal{R}_{1,1}$.

Consequences on correlators:

$$\langle\,\mathbb{I}\,\rangle = 0\,, \qquad \langle\omega(z)\rangle = a\,, \qquad \langle\omega(z)\omega(w)\rangle = -2a\log{(z-w)} + b.$$

More indecomposable reps

Other indecomposable representations $\mathcal{R}_{r,1}$, for $r = 2, 3, 4, \ldots$

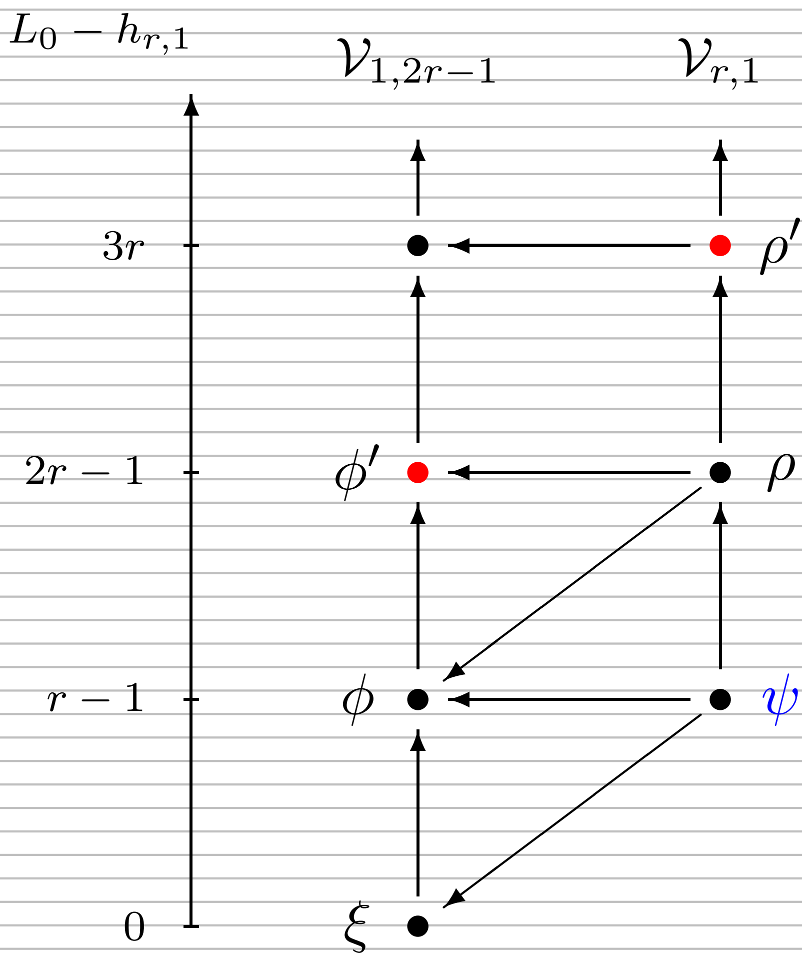
$$L_0\,\psi = h_{r,1}\,\psi + \phi,$$

$$L_p\,\psi = 0, \qquad \forall p > 1$$

$$L_1^{r-1}\,\psi = \beta\,\xi,$$

$$\phi' = [L_{-1}^{2r-1} + \ldots]\,\xi \equiv 0,$$

$$\rho' = [L_{-1}^{2r+1} + \ldots]\,\psi \equiv 0.$$



$\mathcal{V}_{1,2r-1}$ and $\mathcal{V}_{r,1}$ are h.w. subrepresentations of $\mathcal{R}_{r,1}$

[Gaberdiel & Kausch, Rohsiepe]

Fusion closure

The set of irreducible h.w. $\mathcal{V}_{r,s}$ $(s = 1, 2)$ and $\mathcal{R}_{r,1}$ $(r = 1, 2, \ldots)$ is closed under fusion :

$$\mathcal{V}_{r_1,1} \star \mathcal{V}_{r_2,1} = \oplus\, \mathcal{V}_{r,1}, \quad \mathcal{V}_{r_1,1} \star \mathcal{V}_{r_2,2} = \oplus\, \mathcal{V}_{r,2}, \quad \mathcal{V}_{r_1,2} \star \mathcal{V}_{r_2,2} = \oplus\, \mathcal{R}_{r,1}$$

$$\mathcal{V}_{r_1,1} \star \mathcal{R}_{r_2,1} = \oplus\, \mathcal{R}_{r,1}, \quad \mathcal{V}_{r_1,2} \star \mathcal{R}_{r_2,1} = \oplus\, \mathcal{V}_{r,2}, \quad \mathcal{R}_{r_1,1} \star \mathcal{R}_{r_2,1} = \oplus\, \mathcal{R}_{r,1}$$

Remains closed if one adds all reducible $\mathcal{V}_{r,s}$ for all $r, s = 1, 2, \ldots$

For instance

$$\mu \star \mu = \mathcal{V}_{1,2} \star \mathcal{V}_{1,2} = [-1/8] \star [-1/8] = \mathcal{R}_{1,1}$$

$$\mu \star \nu = \mathcal{V}_{1,2} \star \mathcal{V}_{2,2} = [-1/8] \star [3/8] = \mathcal{R}_{2,1}$$

$$\mu \star \mathcal{R}_{2,1} = \mathcal{V}_{1,2} + 2\,\mathcal{V}_{2,2} + \mathcal{V}_{3,2}$$

Warning ...

The set of representations $\mathcal{V}_{r,s}$ and $\mathcal{R}_{r,1}$ is not the complete set of Vir representations for $c = -2$!

Note in particular : fractional weight states remain in irreducible representations, only integral weight states may belong to indecomposables.

However closed under fusion and forms a first natural supply of representations to consider.

For ASM applications, so far, seems enough to account for all known features ...

A lagrangian realization

Simplest and most studied LogCFT.

Precious guide but not realized in ASM ...

$$S = \frac{1}{\pi} \int \partial\theta \bar{\partial}\tilde{\theta} \qquad \text{(symplectic fermions)}$$

- θ and $\tilde{\theta}$ are scalar, anticomm. fields, with canonical dimension 0 $\longrightarrow$ four fields $\mathbb{I}$, θ, $\tilde{\theta}$, $\omega =: \tilde{\theta}\theta :$ of dimension 0, two are bosonic
- Wick contraction $\theta(z,\bar{z})\,\tilde{\theta}(w,\bar{w}) = -\log|z-w|$
- stress-energy tensor $T(z) = -2 :\partial\theta\,\partial\tilde{\theta}: \qquad \longrightarrow \qquad c = -2$
- identity $\mathbb{I}$ and $\omega = :\theta\tilde{\theta}:$ form a Jordan cell (ω is log partner of $\mathbb{I}$)

$$T(z)\omega(w) = \frac{\mathbb{I}}{(z-w)^2} + \frac{\partial\omega}{z-w} + \dots$$

Indecomposable $\mathcal{R}_{1,1}$

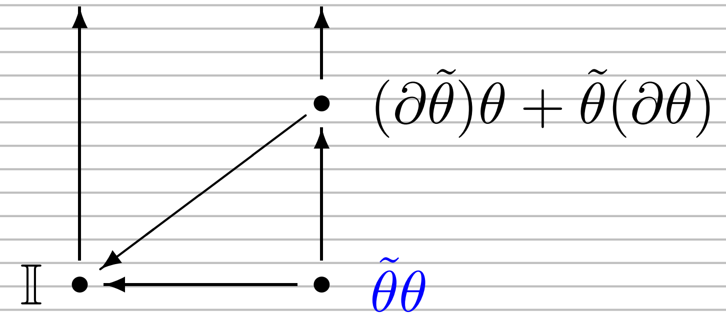
Because of zero modes of $\theta, \tilde{\theta}$ (remember $\int \mathrm{d}\theta_0 = 0$)

$$\langle \mathbb{I} \rangle = 0.$$

However since $\int \mathrm{d}\theta_0 \, \theta_0 = 1$, one has

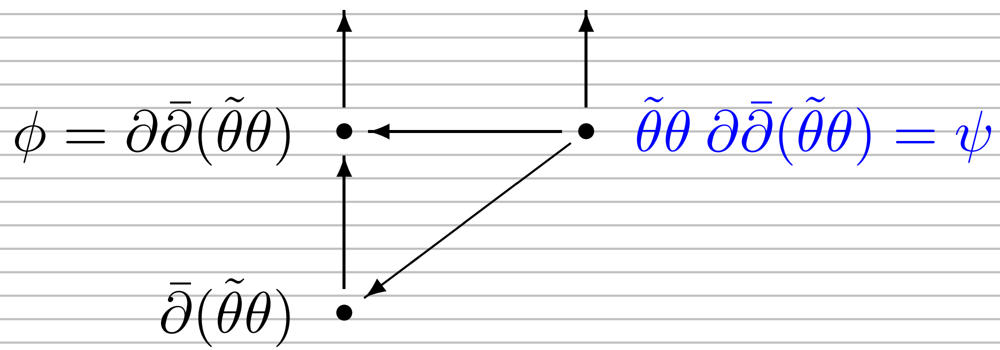
$$\langle \omega(z) \rangle = \langle \tilde{\theta}\theta \rangle = 1, \qquad \langle \omega(z)\omega(w) \rangle = -2 \log |z-w|.$$

The fields $\omega = \tilde{\theta}\theta$ generates an indecomposable (non-chiral) representation $\mathcal{R}_{1,1}$



Indecomposable $\mathcal{R}_{2,1}$

Likewise, the weight 1 field $\psi = \omega\,\partial\bar\partial\omega = \tilde\theta\theta\,\partial\bar\partial(\tilde\theta\theta)$ generates an indecomposable $\mathcal{R}_{2,1}$



Two-point functions read

$$\langle\phi(z)\phi(w)\rangle = 0, \quad \langle\phi(z)\psi(w)\rangle = \frac{a}{(z-w)^2}$$
$$\langle\psi(z)\psi(w)\rangle = \frac{1}{(z-w)^2}\left[-2a\log|z-w| + b\right]$$

Rational LogCFT

The symplectic fermion field theory has an extended symmetry, generated by three weight 3 conserved currents satisfying a W-algebra w.r.t. to which finite number of representations

$$\text{boson}: \ \mathcal{V}_{-1/8},\ \mathcal{R}_0, \qquad \text{fermion}: \ \mathcal{V}_{3/8},\ \mathcal{R}_1$$

So is rational w.r.t. this extended symmetry.

This Lagrangian theory describes many aspects of ASM, but ... not all !!

– Part III –
LogCFT at work : the ASM on the lattice

Testable issues

Following questions involve local lattice observables and should be described by local fields in scaling limit:

1. Correlations of height variables (***)
2. Effect of changing the boundary conditions (**)
3. Effect of introducing additional dissipation (*)

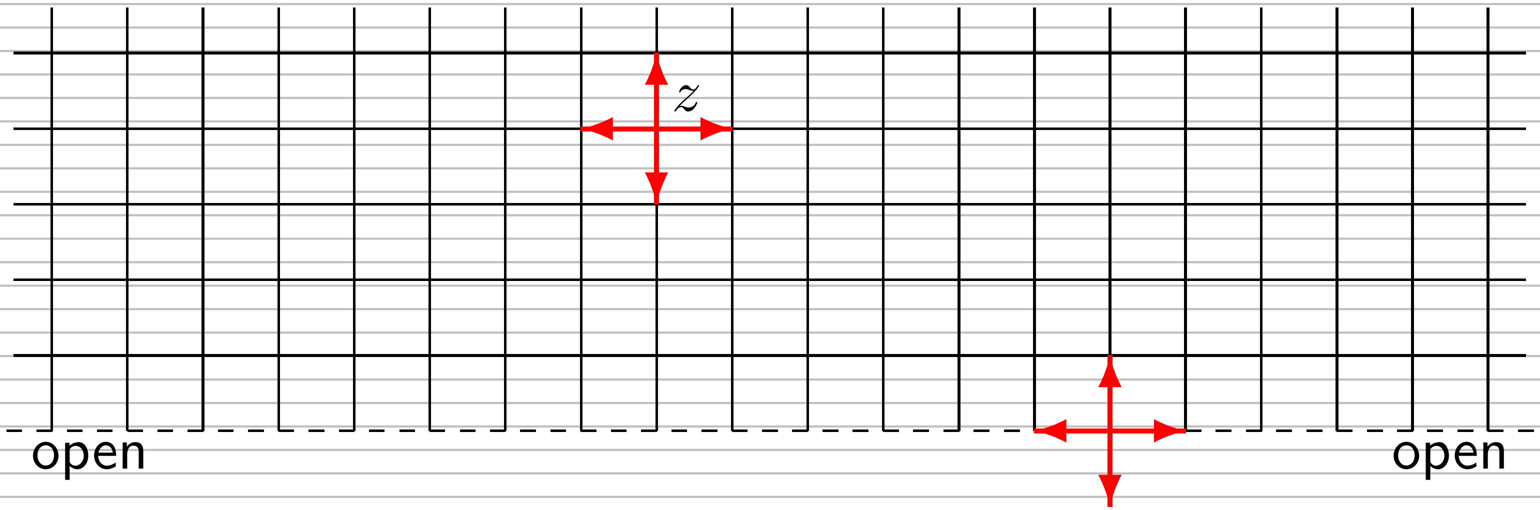
Need correlators in infinite volume.

Here : we take the infinite volume limit of finite volume formulae.

Alternative : first formulate ASM in infinite volume and study stationary measures. [see review by Frank Redig, Les Houches lectures 05]

Dissipation

So far, all sites away from boundaries are conservative.
We decide to introduce dissipation at z, in the bulk of UHP:



So far: $\Delta_{ii} = 4, \quad \Delta_{\langle ij \rangle} = -1$ (loses 4, gives 1 to n.n.)

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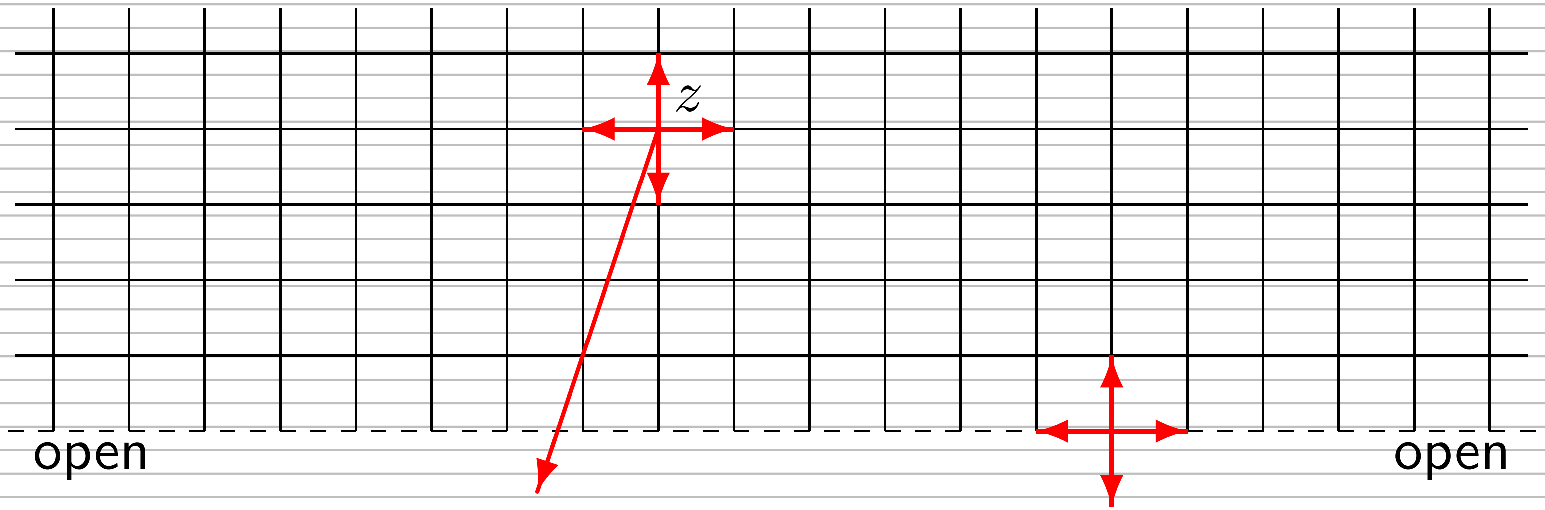


So far: $\Delta_{ii} = 4, \quad \Delta_{\langle ij \rangle} = -1$ (loses 4, gives 1 to n.n.)

Minimal dissipation: $\Delta'_{zz} = 5, \quad \Delta'_{\langle zj \rangle} = -1$ (loses 5, gives 1 to n.n.)

New toppling matrix: $\quad \Delta'_{ij} = \Delta_{ij} + B, \qquad B = \delta_{i,z}\,\delta_{j,z}.$

The effect of introducing dissipation can be measured by the fraction by which the number of recurrent configurations increases:

$$\frac{\det \Delta'}{\det \Delta} = \frac{\#\text{ recurrent configs in new model}}{\#\text{ recurrent configs in original model}}$$

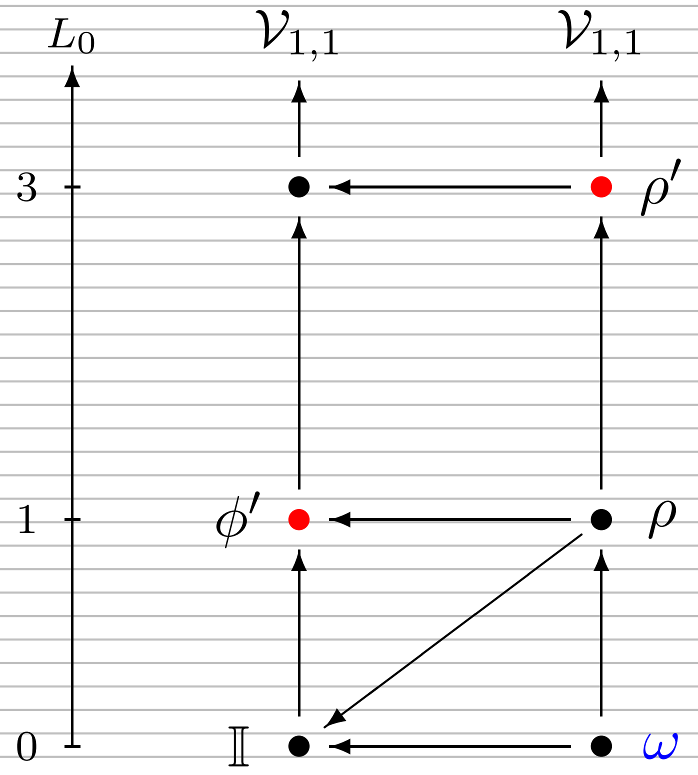
As $B = \Delta' - \Delta$ is a rank 1 perturbation,

$$\frac{\det \Delta'}{\det \Delta} = \frac{\det \Delta + B}{\det \Delta} = \det[(\Delta + B)\Delta^{-1}] = \det[\mathbb{I} + B\Delta^{-1}]$$
$$= 1 + G^{\text{uhp}}_{z,z} = 1 + G^{\text{plane}}_{z,z} - G^{\text{plane}}_{z,\bar{z}}$$
$$= \frac{1}{2\pi}\log|z - \bar{z}| - \gamma_0 + \ldots = \langle \omega(z,\bar{z}) \rangle_{\text{uhp}}$$

where lattice meets CFT

with $\omega(z,\bar{z})$ implementing the insertion of dissipation at z, in SL.

Remember :



Defining relations of $\mathcal{R}_{1,1}$ are:

$$L_0\,\omega = \mathbb{I}, \qquad L_0\,\mathbb{I} = 0,$$

$$L_p\,\mathbb{I} = L_p\,\omega = 0, \qquad \forall p > 0$$

$$\phi' = L_{-1}\,\mathbb{I} \equiv 0,$$

$$\rho' = [L_{-1}^2 - L_{-2}]L_{-1}\,\omega \equiv 0.$$

Left $\mathcal{V}_{1,1}$ is a h.w. subrepres. of $\mathcal{R}_{1,1}$.

Consequences on correlators:

$$\langle\,\mathbb{I}\,\rangle = 0\,, \qquad \langle\omega(z)\rangle = a\,, \qquad \langle\omega(z)\omega(w)\rangle = -2a\log\left(z-w\right) + b.$$

Since

$$\langle \omega(z, \bar{z}) \rangle_{\text{uhp}} = \langle \omega(z) \omega(\bar{z}) \rangle, \qquad \text{(Cardy)}$$

the following identification makes sense :

insertion of isolated dissipation $\longleftrightarrow$ insertion of field $\omega(z, \bar{z}) \in \mathcal{R}_{1,1}$

<u>Checked</u> :

✓ insertion of dissipation at different points

✓ isolated dissipation on a closed boundary $\longrightarrow$ chiral field $\omega(x) \in \mathcal{R}_{1,1}$

✓ dissipation at all sites : system no longer critical (expon. decays)

Pertubation of CFT by $m^2 \int \omega(z, \bar{z}) \sim m^2 \int \tilde{\theta}\theta$ (mass term)

(Realized by fermions)

Turns out that the ω's have a realization in terms of symplectic fermions.

All calculations are exactly compatible with following identifications :

$$\omega_{\text{bulk}}(z,\bar{z}) \equiv \text{(insertion of dissipation at bulk } z) = \frac{1}{2\pi}\theta\tilde{\theta} + \gamma_0\,\mathbb{I}$$

$$\omega_{\text{cl}}(x) \equiv \text{(insertion of dissipation at closed } x) = \frac{1}{2\pi}\theta\tilde{\theta} + (2\gamma_0 - \frac{5}{4})\,\mathbb{I}$$

so that

$$\frac{\det[\Delta + B_1 + \cdots B_n]}{\det\Delta} = \langle\omega(1)\ldots\omega(n)\rangle$$

computed from Wick contractions.

Note: on open boundary, already dissipative, dissipation is less relevant

$$\text{(insertion of dissipation at open } x) = \frac{2}{\pi}\partial\theta\partial\tilde{\theta} \qquad \text{(dim. 2)}$$

Dissipation: summary

The insertion of isolated dissipation at a conservative site

(creation of a bond to sink/root)

corresponds, in the scaling limit, to the insertion of a field ω of weight 0,

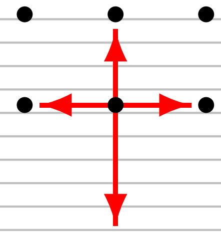
the logarithmic partner of the identity.

The field ω and the identity are the lowest fields in an indecomposable representation $\mathcal{R}_{1,1}$.

Boundary conditions

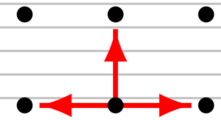
- open boundary site (dissipative)

$$\Delta_{ii} = 4, \quad \Delta_{\langle ij \rangle} = -1,$$



- closed boundary site (conservative)

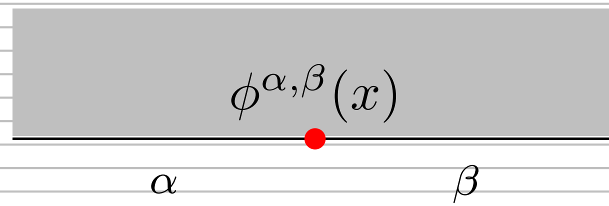
$$\Delta_{ii} = 3, \quad \Delta_{\langle ij \rangle} = -1,$$



- left or right boundary arrows
  Trees are constrained to contain certain boundary bonds

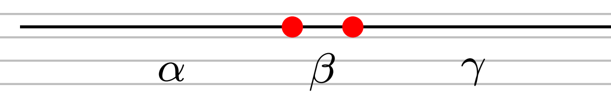
B.c. changing fields

- set $\mathcal{B} = \{\alpha\}$ of conformally invariant b.c.'s
- $\mathcal{B}$ can be finite or infinite (our case)
- a change of boundary condition at a point x, from α to β is realized by the insertion of a (chiral) boundary field $\phi^{\alpha,\beta}$



Also : b.c.c.f. $\phi^{\alpha,\beta}$ are primary fields satisfying a boundary fusion algebra (composition law) with identity $\phi^{\alpha,\alpha} = \mathbb{I}$:

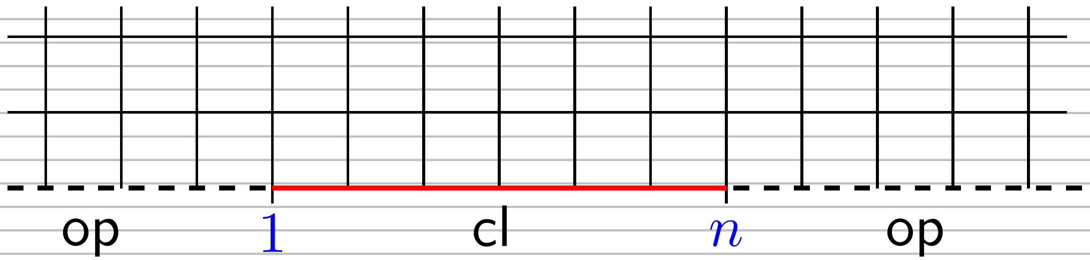
$$\lim_{x \to y} \phi^{\alpha,\beta}(x) \star \phi^{\beta,\gamma}(y) \simeq \phi^{\alpha,\gamma}(y)$$



Assumption : all $\phi^{\alpha,\beta}$ belong to h.w. $\mathcal{V}_{r,s}$ or indecomp. $\mathcal{R}_{r,1}$

Open $\leftrightarrow$ closed

First, well-known case : change from open to closed



The change of boundary condition from open to closed, and vice-versa, is effected, in the scaling limit, by the insertion of a chiral, boundary primary field $\phi^{\mathrm{op,cl}} = \phi^{\mathrm{cl,op}} \equiv \mu$ with conformal dimension $-\frac{1}{8}$.

This primary field belongs to an irreducible representation $\mathcal{V}_{1,2}$.

Fixed arrows

Spanning trees are constrained to contain certain boundary bonds, with the arrow indicating the direction to the root

Same idea as before: insert in an open or in a closed boundary, a string of n consecutive arrows pointing to the left or to the right.

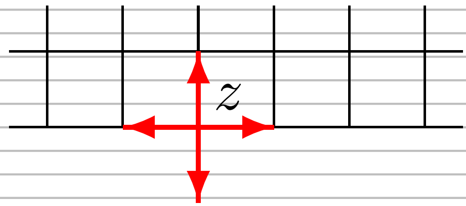
Measure the effect by the ratio:

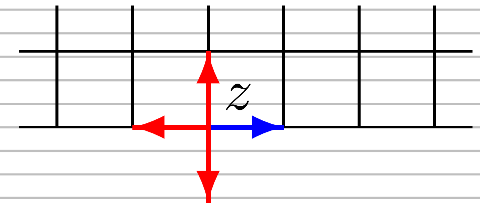
$$\frac{\#\{\text{spanning trees with } n \text{ prescribed arrows}\}}{\#\{\text{spanning trees}\}}$$

Note : left and right arrows are not identical $\longrightarrow$ oriented b.c.'s !

Imposing arrows

Open boundary site





$\Delta^{\text{op}}_{z,\cdot} = (\ldots, -1, 4, -1, -1, 0, \ldots)$ $\qquad$ $\Delta'_{z,\cdot} = (\ldots, -1, 3+\delta, -1, -\delta, 0, \ldots)$

In spanning tree, only one of the four arrows is used: the red arrows bring a weight 1, the blue arrow brings a weight δ:

$$\lim_{\delta\to\infty} \frac{1}{\delta} \det \Delta' = \#\{\text{spanning trees with blue arrow}\}$$

$$\frac{\#\{\text{spanning trees with blue arrow}\}}{\#\{\text{spanning trees}\}} = \lim_{\delta\to\infty} \frac{1}{\delta} \frac{\det\left[\Delta^{\text{op}} + \begin{pmatrix} \delta & -\delta \\ 0 & 0 \end{pmatrix}\right]}{\det \Delta^{\text{op}}}$$

Imposing arrows

Same for closed boundary site

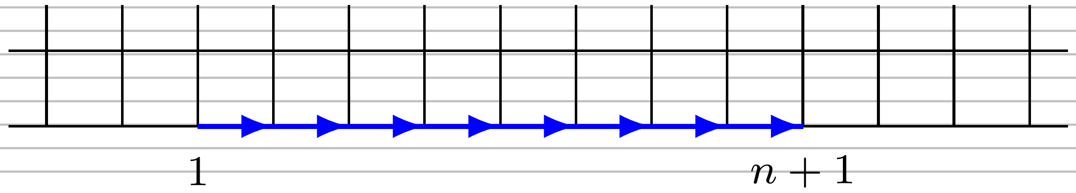
$\Delta^{\mathrm{cl}}_{z,\cdot} = (\ldots, -1, 3, -1, -1, 0, \ldots)$ $\qquad$ $\Delta'_{z,\cdot} = (\ldots, -1, 2+\delta, -1, -\delta, 0, \ldots)$

In spanning tree, only one of the three arrows is used: the red arrows bring a weight 1, the blue arrow brings a weight δ:

$$\lim_{\delta\to\infty} \frac{1}{\delta} \det \Delta' = \#\{\text{spanning trees with blue arrow}\}$$

$$\frac{\#\{\text{spanning trees with blue arrow}\}}{\#\{\text{spanning trees}\}} = \lim_{\delta\to\infty} \frac{1}{\delta} \frac{\det\left[\Delta^{\mathrm{cl}} + \begin{pmatrix} \delta & -\delta \\ 0 & 0 \end{pmatrix}\right]}{\det \Delta^{\mathrm{cl}}}$$

Inserting arrows ...



For n arrows inserted, must compute $(n+1) \times (n+1)$ determinant

$$\lim_{\delta \to \infty} \frac{1}{\delta^n} \frac{\det[\Delta + B]}{\det \Delta}, \qquad B = \begin{pmatrix} \delta & -\delta & 0 & \cdots \\ 0 & \delta & -\delta & 0 \\ 0 & 0 & \delta & -\delta \\ & & \cdots & \end{pmatrix}$$

Little calculation yields

$$\ldots = \det[G_{i,j} - G_{i+1,j}]_{1 \le i,j \le n} = \det(\sigma_{i-j}), \qquad G^{-1} = \Delta^{\rm op} \text{ or } \Delta^{\rm cl}$$

Horizontal invariance $\longrightarrow$ has a Toeplitz form

… in closed

Toeplitz determinants with Fisher-Hartwig singularity. Results are

Closed

Can show

$$\lim_{\delta\to\infty} \frac{1}{\delta^n} \det[\mathbb{I} + G^{\mathrm{cl}} B] = \mathrm{const} \times n^{-1/4}\, \mathrm{e}^{-\frac{2G}{\pi} n} + \dots$$

Involves insertion of two fields $\phi^{\mathrm{cl},\to}(0)$ and $\phi^{\to,\mathrm{cl}}(n)$, and therefore sum of dimensions equal to $-\frac{1}{4} = -\frac{1}{8} + \frac{3}{8}$. In fact :

$\phi^{\mathrm{cl},\to}(0) \equiv \mu'$ has weight $-\frac{1}{8}$, primary irreducible in $\mathcal{V}_{1,2}$

$\phi^{\to,\mathrm{cl}}(n) \equiv \nu$ has weight $\frac{3}{8}$, primary irreducible in $\mathcal{V}_{2,2}$

Important : does not correspond to $\langle \mu'(0)\nu(n)\rangle = 0$ (no dissipation), but to $\langle \mu'(0)\nu(n)\omega(\infty)\rangle = n^{-1/4}$ **with dissipation at** ∞ **!**

Other checks on 3-points and 4-points confirm

$\phi^{\alpha,\beta}$	open	closed	$\longrightarrow$	$\longleftarrow$
open	id.	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$		
closed	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	id.	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	$[\frac{3}{8}] \in \mathcal{V}_{2,2}$
$\longrightarrow$		$[\frac{3}{8}] \in \mathcal{V}_{2,2}$	id.	
$\longleftarrow$		$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$		id.

... in open

Open

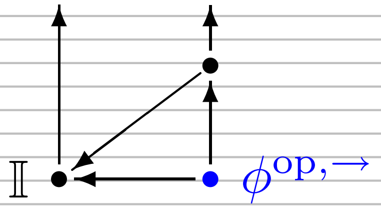
Can show

$$\lim_{\delta\to\infty} \frac{1}{\delta^n} \det[\mathbb{I} + G^{\mathrm{op}} B] = \mathrm{const} \times n^0\, \mathrm{e}^{-\frac{2G}{\pi} n} + \ldots$$

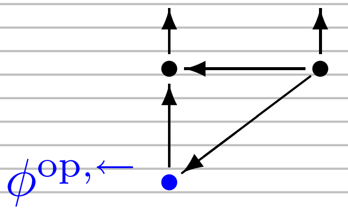
Involves insertion of two fields $\phi^{\mathrm{op},\rightarrow}(0)$ and $\phi^{\rightarrow,\mathrm{op}}(n)$, and therefore sum of dimensions equal to $0 = 0 + 0 \longrightarrow$ both fields have dimension 0.

$\phi^{\mathrm{op},\rightarrow} \in \phi^{\mathrm{op,cl}} \star \phi^{\mathrm{cl},\rightarrow} = \mu \star \mu' = \mathcal{V}_{1,2} \star \mathcal{V}_{1,2} = \mathcal{R}_{1,1}$

goes over to quotient $\mathcal{V}_{1,3} = \mathcal{R}_{1,1}/\mathbb{I}$



$\phi^{\rightarrow,\mathrm{op}} \in \phi^{\rightarrow,\mathrm{cl}} \star \phi^{\mathrm{cl,op}} = \nu \star \mu = \mathcal{V}_{2,2} \star \mathcal{V}_{1,2} = \mathcal{R}_{2,1}$



Other checks on 3-points and 4-points confirm

$\phi^{\alpha,\beta}$	open	closed	$\longrightarrow$	$\longleftarrow$
open	id.	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	$[0] \in \mathcal{V}_{1,3}$	$[0] \in \mathcal{R}_{2,1}$
closed	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	id.	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	$[\frac{3}{8}] \in \mathcal{V}_{2,2}$
$\longrightarrow$	$[0] \in \mathcal{R}_{2,1}$	$[\frac{3}{8}] \in \mathcal{V}_{2,2}$	id.	
$\longleftarrow$	$[0] \in \mathcal{V}_{1,3}$	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$		id.

Other changes

Further calculations of determinants (mainly numerical) yield

$\phi^{\leftarrow,\rightarrow}$ has weight 0

must be in $\phi^{\leftarrow,\mathrm{cl}} \star \phi^{\mathrm{cl},\rightarrow} = \mu' \star \mu' = \mathcal{V}_{1,2} \star \mathcal{V}_{1,2} = \mathcal{R}_{1,1}$

descends to quotient $\mathcal{V}_{1,3}$.

$\phi^{\rightarrow \overset{\mathrm{cl}}{,} \leftarrow}$ has weight 1

must be in $\phi^{\rightarrow,\mathrm{cl}} \star \phi^{\mathrm{cl},\leftarrow} = \nu \star \nu = \mathcal{V}_{2,2} \star \mathcal{V}_{2,2} = \mathcal{R}_{1,1} + \mathcal{R}_{3,1}$

$\phi^{\rightarrow \overset{\mathrm{op}}{,} \leftarrow}$ has weight 0

in $\phi^{\rightarrow,\mathrm{op}} \star \phi^{\mathrm{op},\leftarrow} = \mathcal{R}_{2,1} \star \mathcal{R}_{2,1} = 2\mathcal{R}_{1,1} + 2\mathcal{R}_{2,1} + 2\mathcal{R}_{3,1} + \mathcal{R}_{4,1}$

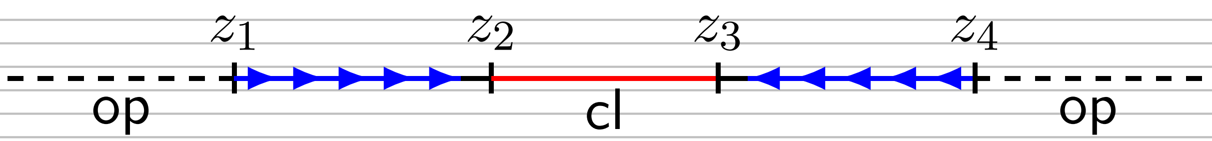
(most probably, deserves further checks)

Boundary conditions: summary

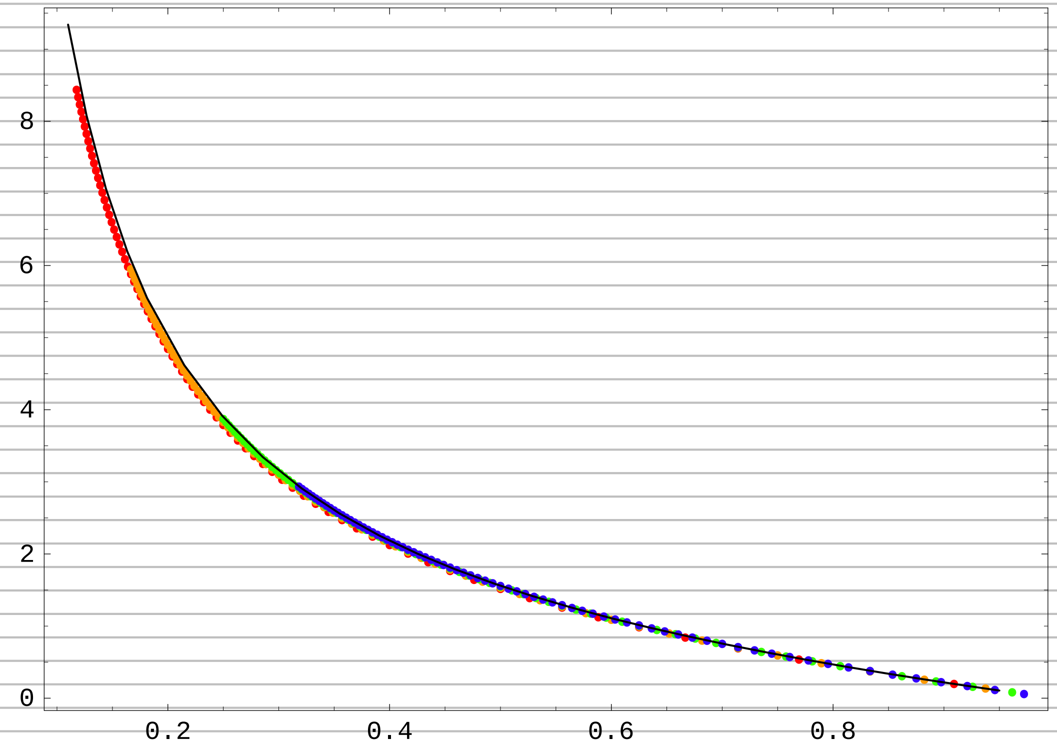
Leads to following table (in present understanding)

$\phi^{\alpha,\beta}$	open	closed	$\longrightarrow$	$\longleftarrow$
open	id.	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	$[0] \in \mathcal{V}_{1,3}$	$[0] \in \mathcal{R}_{2,1}$
closed	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	id.	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	$[\frac{3}{8}] \in \mathcal{V}_{2,2}$
$\longrightarrow$	$[0] \in \mathcal{R}_{2,1}$	$[\frac{3}{8}] \in \mathcal{V}_{2,2}$	id.	$[0] \in \mathcal{R}_{2,1}$ (op) $[1] \in \mathcal{R}_{3,1}$ (cl)
$\longleftarrow$	$[0] \in \mathcal{V}_{1,3}$	$[-\frac{1}{8}] \in \mathcal{V}_{1,2}$	$[0] \in \mathcal{V}_{1,3}$	id.

Cross-checks



Corresponds to $\langle \sigma(1)\nu(2)\nu(3)\sigma(4)\rangle = \beta\ z_{23}^{-3/4}\ \frac{1-x}{\sqrt{x}}$



Height variables

Most natural but hardest !

Purpose = compute joint probas $P^*[h_{z_1} = a,\, h_{z_2} = b,\, \ldots]$

Plane 1-point probas computed in '91 (height 1; Dhar & Majumdar) and in '94 (heights 2,3,4; Priezzhev), but are ignored by the FT description:

$$P^*(a) = P^*[h_z = a] = \langle \delta(h_z - a) \rangle_{P^*} \neq 0 \qquad \longleftrightarrow \qquad \langle h_a(z) \rangle = 0$$

As FT describes correlation functions, the proper correspondence reads

$$\delta(h_z - a) - P^*(a) \qquad \longleftrightarrow \qquad \text{field } \; h_a(z)$$

under which

$$\text{scalim} \; \Big\{ P^*[h_{z_1} = a,\, h_{z_2} = b] - P^*(a)\, P^*(b) \Big\} = \langle h_a(z_1)\, h_b(z_2) \rangle$$

Height variables

The identification of scaling fields h_a requires computing lattice correlation functions of height variables ...

Fine for heights 1 (boundary or bulk)

More difficult for heights 2,3,4 on boundary (open or closed)

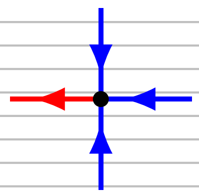
Still harder for heights 2,3,4 in bulk !

Why ??

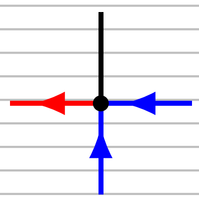
Trees, branches, leaves

Need spanning tree description of recurrent configurations of ASM.

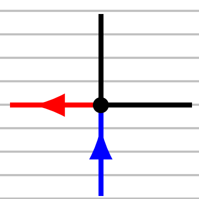
Remember the burning algorithm, building the spanning tree:



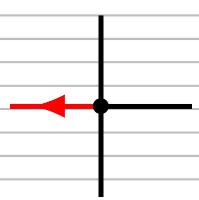
height can only be equal to 4: $P_4 = P_3 + \frac{\mathcal{N}_3}{\mathcal{N}}$



height can be equal to 3 or 4: $P_3 = P_2 + \frac{\mathcal{N}_2}{2\mathcal{N}}$



height can be equal to 2, 3 or 4: $P_2 = P_1 + \frac{\mathcal{N}_1}{3\mathcal{N}}$



height can be equal to 1, 2, 3 or 4: $P_1 = \frac{\mathcal{N}_0}{4\mathcal{N}}$

Predecessors

Previous formulae require computing the number of trees with fixed number of predecessors at given site z:

$$\mathcal{N}_k = \text{number of configs such that } z \text{ has set fire to exactly } k \text{ n.n.}$$

Huge difference between $k = 0$ and $k > 0$:

$\mathcal{N}_0$ is local: reference site is a leaf; local constraint

$\mathcal{N}_{k>0}$ is non-local: must exclude big fire path in lattice which eventually comes back to a nearest neighbour; non-local constraint

Heights 1 are easier, while heights 2, 3, 4 are harder !!

1-site probabilities

Can see it on the answers:

$$P_1 = \frac{2(\pi - 2)}{\pi^3} = 0.0736$$

$$P_2 = \frac{1}{2} - \frac{1}{\pi} - \frac{3}{\pi^2} + \frac{12}{\pi^2} - \frac{\pi - 2}{2\pi} J_2$$

with

$$J_2 = \frac{4}{\pi^2} - \frac{14}{\pi} - 8 - \frac{4\sqrt{2}}{\pi^2} \int_0^{\pi} \frac{\mathrm{d}\beta_1}{\sqrt{3 - \cos\beta_1}} \int_{-\pi}^{\pi} \frac{\mathrm{d}\beta_2}{1 - t_1 t_2 t_3} \sin\frac{\beta_1 - \beta_2}{2}$$
$$\left[\cos\frac{\beta_1 - \beta_2}{2} - 2\cos\frac{\beta_1 + \beta_2}{2}\right] \times \left[(3 - \cos\beta_1 + \cos\beta_2)\cos\frac{\beta_1}{2} - 2\sin\beta_2 \sin\frac{\beta_1}{2}\right],$$

where $t_i = y_i - \sqrt{y_i^2 - 1}$, $y_i = 2 - \cos\beta_i$ and $\beta_3 = -(\beta_1 + \beta_2)$.

Remarkably $J_2 = 0.5 + o(10^{-12})$, but no proof !

1-site probabilities

Can see it on the answers:

$$P_1 = \frac{2(\pi - 2)}{\pi^3} = 0.0736$$

$$P_2 = \frac{1}{2} - \frac{1}{\pi} - \frac{3}{\pi^2} + \frac{12}{\pi^2} - \frac{\pi - 2}{2\pi} J_2 = 0.1739$$

$$P_3 = \frac{1}{4} + \frac{2}{\pi} - \frac{12}{\pi^3} - \frac{8 - \pi}{4\pi} J_2 = 0.3063$$

$$P_4 = 1 - P_1 - P_2 - P_3 = 0.4461$$

Note $P_1 < P_2 < P_3 < P_4$ in agreement with forbidden subconfigurations picture.

Higher heights

On UHP, compute 1-site probability to have height 2,3,4 at a distance m from boundary, open or closed.

Asymptotic analysis for m large yields dominant contributions in SL :

$$P_i^{\rm op}(m) = P_i + \frac{1}{m^2}(a_i + \frac{b_i}{2} + b_i\,\log m) + \ldots,$$
$$P_i^{\rm cl}(m) = P_i - \frac{1}{m^2}(a_i + b_i\,\log m) + \ldots,$$

with explicit coefficients,

$$a_1 = \frac{\pi-2}{2\pi^3}, \qquad b_1 = 0$$
$$a_2 = \frac{\pi-2}{2\pi^3}\Big(\gamma + \frac{5}{2}\log 2\Big) - \frac{11\pi-34}{8\pi^3}, \qquad b_2 = \frac{\pi-2}{2\pi^3}$$
$$a_3 = \frac{8-\pi}{4\pi^3}\Big(\gamma + \frac{5}{2}\log 2\Big) + \frac{2\pi^2+5\pi-88}{16\pi^3}, \qquad b_3 = \frac{8-\pi}{4\pi^3}$$

Bulk heights: summary

1-site probability on UHP is a disguised (chiral) 2-pt correlation (image), and allows the field identification.

All checks confirm that :

The height 1 field h_1 is a primary field with weights (1,1).

The others three h_2, h_3, h_4 also have weights (1,1), and are equal, up to normalizations, to the same field, the logarithmic partner of h_1.

The four fields h_i belongs to a non-chiral indecomposable $\mathcal{R}_{2,1}$.

Conclusions

Good number of features well understood:

- 4 boundary conditions identified, leading to b.c. changing fields with conformal weights $0, -\frac{1}{8}, \frac{3}{8}, 1$
- isolated dissipation, on boundary or in bulk, with and without change of b.c.; bulk, boundary and bulk-boundary fusions checked
- boundary height variables on closed and open boundaries (not log)
- bulk height variables properly identified (log fields), with and without change of b.c.
- fully dissipative model, no longer critical, described by massive perturbation of $c = -2$

Open issues:

- relevant LCFT likely to be non-rational: complete its identification
- look for other boundary conditions
- identify new bulk obervables
- establish relationships with other models

Perspective:

Avalanche observables, SLE ?